THE USE OF THE INTEGRAPH IN THE PRACTICAL SOLUTION OF DIFFERENTIAL EQUATIONS BY PICARD'S METHOD OF SUCCESSIVE APPROXIMATIONS

BY DR. THORNTON C. FRY,


INTRODUCTION

1. Many engineering problems are easily formulated in terms of differential equations. Frequently these are of a simple type and easily solved, although this is by no means universally true. When they are more difficult they can generally be solved by the use of suitable series expansions, but this entails labour and expense which may not be justified by the economic importance of the information obtained.

Bessel's equation may serve as an illustration. It has no simple solution in terms of the elementary functions, and if it were necessary to start from the beginning and develop a solution without reference to past studies, the cost might very well be prohibitive and the problem might have to go unsolved.

There are many equations, inherently no more difficult than Bessel's, which are of less general interest and have not been studied so extensively. If such equations are to be solved at all, so far as industrial mathematics is concerned, some simple method of procedure must be developed for the purpose. It is not necessary that this method yield absolute accuracy, for few of the magnitudes with which the engineer has to deal are more than approximations. It is sufficient for it to compare favourably with the best standards of engineering accuracy.

The method explained in this paper has been found valuable in treating problems of this kind. It is founded upon the Picard method of successive approximations, from which it differs only in certain minor details.

Theoretically, therefore, it is capable of yielding any desired degree of accuracy, if the computer is persistent enough; but practically it requires the use of some form of mechanical integration, and the accuracy of the results is therefore limited by that of the mechanical device.

What means of integration is used is a matter to be decided on the basis of convenience. From the standpoint of speed the most satisfactory results are obtained with the integraph, and as the errors thus produced are comparable with the inaccuracies of curve plotting they are sufficiently small to be unimportant in most phases of applied mathematics.
2. Suppose, to start with, that the problem presents itself, not as a single equation defining a single function \( y \), but as a set of \( n \) equations defining \( n \) functions, \( y, y', \ldots, y^{(n-1)} \). Suppose, moreover, that all these equations are of the first order*, so that they may be written in the form

\[
\frac{dy}{dx} = f(x, y, y', \ldots, y^{(n-1)}),
\]

\[
\frac{dy'}{dx} = f'(x, y, y', \ldots, y^{(n-1)}),
\]

\[
\frac{dy^{(n-1)}}{dx} = f^{(n-1)}(x, y, y', \ldots, y^{(n-1)}),
\]

and that the desired solution is required to satisfy the conditions

\[
y = \eta, y' = \eta', \ldots, y^{(n-1)} = \eta^{(n-1)},
\]
at \( x = \xi \).

If the true solution of these equations were known, it would be a set of \( n \) explicit functions of \( x \); and if all the \( y \)'s in the right-hand members of (1) were replaced by these known functions, the equations would reduce to an exceedingly simple form, in which the \( x \)-derivatives of each of the \( y \)'s would be given directly as functions of \( x \). The equations could then be integrated directly, and made to satisfy the boundary conditions. Formally, the result would be:

\[
y = \eta + \int_{\xi}^{x} f(x, y, y', \ldots, y^{(n-1)}) \, dx,
\]

\[
y' = \eta' + \int_{\xi}^{x} f'(x, y, y', \ldots, y^{(n-1)}) \, dx,
\]

\[
y^{(n-1)} = \eta^{(n-1)} + \int_{\xi}^{x} f^{(n-1)}(x, y, y', \ldots, y^{(n-1)}) \, dx.
\]

*This restriction is, of course, largely a formal one, for it is possible to write any set of ordinary differential equations in the form of a set of first order equations. Thus, the equation

\[
\frac{d^2y}{dx^2} + \frac{dy}{dx} + \left[ \frac{dy}{dx} \right]^2 + 2y - x = 0,
\]

may be thrown into the form

\[
\frac{dy}{dx} = y', \quad \frac{dy'}{dx} = y'', \quad \frac{dy''}{dx} = x - 2y - y^2 - xy',
\]

which is a special case of (1); and a similar transformation can obviously be applied to a set of equations as well. The generality of (1) is therefore very great.
This manipulation seems to have resulted in nothing new, for the y's defined by (3) are exactly the same as the y's that occur in the integrands. If they are known, nothing has been accomplished; if they are not known, no means of finding them has been presented. Nevertheless, the equations (3) are of value, in that they afford a means of checking a set of trial functions to determine whether or not they satisfy (1). For if these trial functions are substituted in the right-hand members of (3), and the indicated integrations are performed, a set of y's will be obtained which may or may not be identical with the trial functions. If they are, the trial functions are indeed the solution of (1); otherwise they are not.

The equations (3) are useful in another and less obvious respect, in that they make it possible to obtain from an incorrect set of trial functions a second set which is better than the first. As this second set can be similarly used in deducing a third, and so on, the process can be continued until it leads to a result which is so nearly identical with the true solution that the differences may be ignored. This is the essential idea back of the Picard method of successive approximation.

For the first approximation the functions y, y', . . ., y^{(n-1)} are assumed to be constants, and, in order that the boundary conditions may be satisfied, these constants are chosen as \( \eta, \eta', \ldots, \eta^{(n-1)} \), respectively. These first approximations are then substituted in the right-hand members of (3), and the indicated integrations are performed, giving a new set of functions,

\[ y_0, y'_1, \ldots, y_{n-1}^{(n-1)}, \]

which may or may not be identical with the y's. If it is, it follows that the \( \eta \)'s were indeed the correct solution, and there is no need of further manipulation. But if not, the \( \eta \)'s are incorrect, and they are therefore abandoned forthwith.

The \( y_2 \)'s are then themselves adopted as a second approximation and substituted in the right-hand side of (3). When the integrations have been performed, still another set of functions, \( y_0, y_0', \ldots, y_{3}^{(n-1)} \), appears. This is the third approximation. Continuing the process, set after set of approximations is deduced, each related to the preceding one by the equations

\[
\begin{align*}
    y_{k+1} &= \eta + \int_{\xi}^{x} f(x, y_{k}, y'_{k}, \ldots, y_{k}^{(n-1)}) \, dx, \\
    y'_{k+1} &= \eta' + \int_{\xi}^{x} f'(x, y_{k}, y'_{k}, \ldots, y_{k}^{(n-1)}) \, dx, \\
    \vdots \\
    y_{k+1}^{(n-1)} &= \eta^{(n-1)} + \int_{\xi}^{x} f^{(n-1)}(x, y_{k}, y'_{k}, \ldots, y_{k}^{(n-1)}) \, dx.
\end{align*}
\]

It may happen—and sometimes does happen—that one of these sets is the correct solution. If so, the process of obtaining one from the other is such that all succeeding sets are identical. In other words, the process has terminated in the correct solution. This outcome, however, is unusual. Ordinarily each
set differs from all those which preceded it; but it can be proved that they differ progressively less and less from one another, and ultimately approach a set of limiting values which is itself the true solution of equations (1). In other words, by starting from the preliminary guess that \( y, y', \ldots, y^{(n-1)} \) are all constant, and by following a well-established routine involving only the integration of known functions of \( x \), a result may ultimately be obtained which differs as little as we please from the desired solution of the set of equations.

This process is capable of graphical interpretation, as is illustrated in Fig. 1 for a set of two equations defining the two functions \( y \) and \( y' \). The first approximation—in which \( y \) and \( y' \) are assumed to be constant and equal to the boundary values 1 and 0—is represented by the straight lines \( y_1 \) and \( y'_1 \) in the upper half of the figure. When these constants are substituted in \( f \) and \( f' \), they give the pair of functions \( f_2 \) and \( f'_2 \) plotted in the lower half of the figure. The second approximations \( y_2 \) and \( y'_2 \) (shown in the upper half of the diagram) are the integrals of these functions, the constants of integration having been so chosen that the boundary values are again satisfied. These second approximations in turn are substituted in the functions \( f \), and lead to a new set of curves \( f_3 \) and \( f'_3 \) which appear in the lower half of the diagram.

Thus the process continues. From each set of \( f \)-curves a pair of \( y \)-curves is obtained by a straightforward process of integration, and then the next set of \( f \)-curves is found by substituting the new \( y \)'s in the right-hand members of (1).

Theoretically there is nothing objectionable about this form of solution. Practically it assumes that the integrations can all be performed, which is unfortunately not often the case. That is, although there is usually no difficulty in passing from a set of \( y \)-curves to the set of \( f \)-curves determined by it, the process of passing from this latter to the next succeeding set of \( y \)-curves is often too
tedious to be feasible. If by some means this objection can be overcome, the Picard method can be made to serve a useful purpose in obtaining numerical results, as well as in theoretical studies.

In those cases where graphical representation is possible—which means that the equations must be reducible to a form which does not contain unknown parameters—the integraph affords a satisfactory solution of this difficulty. This instrument, which is shown in Fig. 2, is in a sense a planimeter, but it differs from ordinary planimeters in two respects: it performs its integrations in the rectangular coordinate system, and it actually traces the integral curve by means of a ruling pen attached to the machine. Thus, if the tracing point were caused to follow the curve $f_2$ of Fig. 1, the pen—having been initially placed upon the point specified by the boundary conditions—would actually draw the integral curve $y_2$.

The instrument is easily controlled, is not difficult to adjust, and seldom departs from the true solution by as much as the width of the ink line which it draws. Its accuracy is therefore sufficient to meet most practical requirements.

Example I. An Illustration of the Use of the Integraph in Picard's Method

3. The curves reproduced in Fig. 1 were obtained by this method of procedure. They are solutions of the set of equations:

$$\frac{dy}{dx} = y', \quad \frac{dy'}{dx} = \left(1 - \frac{1}{x^2}\right)y - \frac{1}{x}y',$$

subject to the boundary conditions

$$y = 1, \quad y' = 0,$$

at $x = 1$.

It therefore follows, from comparison of (4) with (1), that the functions $f$ are

$$f(x, y, y') = y', \quad f'(x, y, y') = \left(1 - \frac{1}{x^2}\right)y - \frac{1}{x}y'.$$
The first set of trial curves, in which \( y \) and \( y' \) are constants, is denoted by \( y_i \) and \( y_i' \). Upon substitution of these constants in (5) the functions \( f \) take the forms

\[
f_2 = 0, \quad f_2' = \frac{1}{x^2} - 1.
\]

They are plotted in the lower half of the figure.

It so happens that these functions could be readily integrated without mechanical aid. In fact, in the case of this particular illustration several approximations could be obtained before the integrations become arduous. Nevertheless in the construction of Fig. 1 all the \( y \) and \( y' \) curves were drawn by the integraph; for it was easier to produce them in that way than to make the computations incidental to plotting them. In particular, \( y_2 \) and \( y_2' \) are the integrals obtained by tracing \( y_2 \) and \( y_2' \).

To obtain \( f_3 \) and \( f_3' \) these new functions \( y_2 \) and \( y_2' \) must be substituted in (5). This is done by reading values from the curves and substituting them in (5), continuing the process for as many values of \( x \) as are necessary to permit the \( f \)-curves to be drawn in. These new \( f \)-curves, \( f_3 \) and \( f_3' \), are then integrated to give \( y_3 \) and \( y_3' \), from which in turn \( f_4 \) and \( f_4' \) are deduced by substitution.

This cycle is repeated again and again, until two consecutive sets of \( y \)'s are so nearly identical as to be indistinguishable on the scale of the drawing. When this stage has been reached, the last curves are accepted as the solution of (4).

As an illustration of the amount of labour required to carry out this process—the complete set of computations is reproduced in Table I. It was first decided to extend the solution as far as \( x = 3 \), and the tabular interval 0.2 was fixed upon as a suitable one from the standpoint of curve plotting. Then the factors \( \frac{1}{x^2} - 1 \) and \( \frac{1}{x} \), by which \( y \) and \( y' \) are multiplied in (5), were computed and recorded in columns 2 and 3, in order that they might be readily available for the work that was to follow.

As \( f_2' \) is equal to \( \frac{1}{x^2} - 1 \), which has already been recorded, and \( f_2 = 0 \), it was not necessary to give these numbers space in the table. Also, \( y_2 \) being constant and equal to 1, there was no need of recording it. But \( y_2' \) is variable, and its values were therefore entered in column 4.

By equation (5), \( f_2 = y_2' \). Hence, the graph of \( f_2 \) was obtained by plotting the numbers given in column 4. To get \( f_2' \) it was necessary to carry out the processes indicated in the second of equations (5): multiplying corresponding entries in columns 3 and 4, and subtracting them from column 2. The results are shown in column 5, and when plotted give the curve \( f_2' \).

The other columns up to 21 are obtained in a similar fashion, those headed \( f' \) being computed from the next preceding \( y \) and \( y' \), and those headed \( y \) or \( y' \) being read directly from the curves of Fig. 1. When \( y_8 \) had been obtained it was obvious that no further improvement was possible. Therefore \( y_8 \) was accepted as the true solution.
TABLE I

COMPUTATIONS INCIDENTAL TO THE SOLUTION OF EXAMPLE I

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To check the accuracy of the result, it was noted that when the relationship \( \frac{dy}{dx} = y' \) is substituted in the second of equations (4), the set reduces to Bessel's Equation

\[
\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left( 1 - \frac{1}{x^2} \right)y = 0,
\]

the solution of which, subject to the assigned boundary conditions, is

\[ y = 1.4034 J_1(x) - 0.3251 Y_1(x). \]

The correct values of \( y \), as computed from this equation, are recorded in column 22. They make it possible to determine the accuracy of the graphical computation, which turns out to have been nowhere in error by more than three units in the third decimal place. This represents less than one fiftieth of an inch on the scale of the original drawing.

**Example II. Illustrating an Alternative Form of Procedure When the Equations Are Not of the First Order**

4. There is one outstanding peculiarity about Fig. 1. It is that the diagrams for \( f \) and \( y' \) are—as a whole—identical. This comes about through the fact noted in the last paragraph, that the set is equivalent to a single equation of higher order, which fact manifests itself in the simple form of the first of equations (4).

Because of this peculiarity, each \( y \)-curve is obtained by double integration of an \( f' \), the first integration giving a \( y' \), and the second a \( y \). But as the cycle of operations now stands, this \( y \) and \( y' \) do not belong to the same set of approximations. It is not \( y_k' \) and \( y_k \) which are the successive integrals of \( f_k' \), but \( y_{k+1}' \) and \( y_{k+1} \) instead. In other words, although equations (4) are of such a form that the symbol \( y' \) must necessarily stand for \( \frac{dy}{dx} \), the \( y' \) which is assigned to a particular set of approximations is not the derivative of the \( y \) which belongs to the same set.

It would be more elegant to have each set of curves stand in their normal differential relations with respect to one another, an improvement which is easily brought about if the first integral of \( f_k' \) is denoted by \( y_k' \), and its second integral by \( y_k \) (not \( y_{k+1} \)). These approximations can then be used in exactly the same way as were those previously denoted by the same symbols. But in so doing, the traditional cycle is modified, and as the classical proofs of the convergence of the process depend upon this traditional cycle, the question arises as to whether the new cycle, like the old, can be relied upon to lead to a solution. The answer is that it may, for it is a comparatively simple exercise in pure mathematics to so revise the classical argument as to apply to the modified procedure. Moreover, it is a matter of experience that such problems as are met in engineering practice can frequently be solved in this way with less labour than would be occasioned by the classical procedure.
To compare this modified cycle with the original one equation (6) has been solved by the revised method. The results are shown in Fig. 3. The approximation curves are obviously different, but they lead to substantially the same result in the end.

Each $f$-curve (as for instance $f_3$) is computed from the preceding set of $y$'s (in this case, $y_2$ and $y_3'$); and from it the next set of $y$'s ($y_3$ and $y_3''$) is obtained by double integration. The computations are reproduced in Table II. The accuracy of the results obtained by this method is about the same as by the original one, and two less approximations are required. When the equation is of high order, the advantage in favour of the modified method is usually even more pronounced.

Fig. 3

It will be noted that the earlier approximations in Fig. 3 are not extended all the way to $x=3$. The reason is, that the inaccuracy of the earlier approximations in the vicinity of $x=3$ is so great that the $f'$-values deduced from them are grossly incorrect. The labour of computing them is therefore virtually wasted. If, instead of dealing with the entire interval at the start, it is partitioned off into a number of sub-intervals, all but the first can be ignored until the solution has been completed over this limited range. The values of $y$ and $y'$ at the end of the first sub-interval then form a new set of boundary values, by means of which the solution may be extended to the second interval. In this way, one sub-interval after another may be dealt with, until the desired range of values is obtained.

If this routine is strictly adhered to, the first approximation in the second region will be a set of constants. But when the straight lines corresponding to these constants are joined to the correct curves already found, they are likely to be obviously incorrect: so obviously incorrect, in fact, that the computer will have no doubt as to his ability to draw a better approximation free hand. Under these circumstances, the labour is simplified by making the best guess possible, and using it instead of the straight lines required by the classical scheme. More-
### TABLE II

Computations Incidental to the Solution of Example II

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\frac{1}{x^2} - 1 = f'_1$</th>
<th>$\frac{1}{x}$</th>
<th>$y'_1$</th>
<th>$\gamma_1$</th>
<th>$f'_2$</th>
<th>$\gamma'_2$</th>
<th>$\gamma_2$</th>
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<table>
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<tr>
<th>$x$</th>
<th>$f'_1$</th>
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<th>$\gamma_1$</th>
<th>$y_2$ (calculated)</th>
<th>error</th>
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over, this procedure is justified by the fact that the straight lines themselves are pure guesses, the use of which has no other foundation than the a posteriori fact that it leads to a solution.

In practice it is generally advisable to make such a guess, not when the first interval has finally been disposed of as accurately as possible, but shortly before. By so doing, the same steps of the process which serve to eliminate the last small errors from the one part of the curves also partially correct the grosser errors in the operator's guess.

It is by this means that the curves of Fig. 3 were obtained. Thus \( f \) was extended from \( x = 2.4 \) to 2.6, and \( f \) to 3.0, by guess, although the correct result at 2.4 was not obtained until the last approximation had been completed.

**Example III. A Problem in Electrical Circuit Theory, Illustrating the Solution of Simultaneous Differential Equations of the Second Degree**

5. Sometimes the modified procedure explained in §4 is applicable even when the set of equations is not reducible to a single equation of higher order. An illustration may be obtained from the consideration of the electrical circuit shown in Fig. 4.

![Fig. 4](image)

If \( x_1 \) represents the instantaneous charge on the condenser \( C_1 \), and \( x_1 - x_2 \) the instantaneous charge on \( C_2 \), the differential equations controlling the behaviour of the circuit are

\[
L_1 \frac{d^2x_1}{dt^2} + R_1 \frac{dx_1}{dt} + \left[ \frac{1}{C_1} + \frac{1}{C_2} \right] x_1 - \frac{1}{C_2} x_2 = E, \tag{7}
\]

\[- \frac{1}{C_1} x_1 + L_2 \frac{d^2x_2}{dt^2} + R_3 \frac{dx_2}{dt} + \frac{1}{C_2} x_2 = 0.\]

Such a set of equations can be rewritten in terms of first order equations by introducing new variables in place of \( \frac{dx_1}{dt} \) and \( \frac{dx_2}{dt} \); and in this instance this
happens to have physical significance, since these derivatives are the currents $I_1$ and $I_2$, respectively. The four equations which result are:

$$\frac{dx_1}{dt} = I_1, \quad L_1 \frac{dI_1}{dt} = E - R_1 I_1 - \left( \frac{1}{C_1} + \frac{1}{C_2} \right) x_1 + \frac{1}{C_2} x_2,$$

(8)

$$\frac{dx_2}{dt} = I_2, \quad L_2 \frac{dI_2}{dt} = \frac{1}{C_1} x_1 - R_2 I_2 - \frac{1}{C_2} x_2.$$

To the equations in this form the Picard method might be applied directly; but if this were done the currents assigned to any one approximation would not be the derivatives of the corresponding charges, as they must be because of a fundamental physical law. It is therefore more satisfactory to deal with the equations in the original form (7) instead of transforming them into (8).

Suppose for example, that the constants have the values

$$L_1 = 0.1, \quad L_2 = 1,$$

$$R_1 = 20, \quad R_2 = 100,$$

$$C_1 = 10^{-5}, \quad C_2 = 10^{-6},$$

and suppose that $x_1$ and $x_2$ are replaced by $x$ and $y$ respectively, in order that no confusion may arise as to the significance of the subscripts. Then (7) reduces to the form

$$x'' = 10 E - 200 x' + 10^6 x - 2.10^6 x,$$

(9)

$$y'' = 10^5 x' - 10^5 y - 100 y',$$

the primes denoting differentiation.

The boundary values, which afford the starting point for the solution, are determined by physical conditions. If it is assumed that no electromotive force is impressed on the circuit prior to $t = 0$, they are

$$x = 0, \quad x' = 0, \quad y = 0, \quad y' = 0.$$

As usual these boundary values are taken as the first approximations, and called $x_1, x_1', y_1, y_1'$. (See Fig. 5.) When substituted in (9) they give

$$x_2'' = 10 E,$$

from which the second approximations $x_2'$ and $x_2$ are obtained by double integration; and also

$$y_2'' = 0,$$

from which $y_2'$ and $y_2$ are similarly obtained. These are again substituted back in (9), and yield $x_3''$ and $y_3''$, the entire cycle being identical with that of Example II, except in so far as it involves two variables. All these curves are shown in Fig. 5.
It should be noticed that it is not necessary to have an algebraic definition of $E$ in order to carry out these processes. A graphical representation—such, for instance, as an oscillograph record or as the curve for $10E$ shown in Fig. 5—serves the purpose just as well, and is even more convenient than an algebraic definition which is hard to compute.

This freedom from the necessity of algebraic definition is inherent in graphical work, and extends to every phase of it. It is not even necessary that the fundamental equations (1) themselves be expressible algebraically; any form of definition by means of which the numerical values of the $f$'s can be obtained when the numerical values of their arguments are known is equally satisfactory. In the case of one variable, for example, the equation would appear as

$$\frac{dy}{dx} = f(x, y),$$

but the symbol $f(x, y)$ need not connote any algebraic law. Instead, the func-
tional relationship may be given by some form of curve, or even by a more or less extensive double-entry table such as might be obtained from laboratory observation. If this table permits the numerical value of the derivative to be obtained when the numerical values of the arguments are given, it defines the functional relationship sufficiently well for the purposes of the problem.

The electromotive force assumed in the construction of Fig. 5 illustrates this point. It was sketched freehand and is representable by no known formula, yet the labour of solving the problem was no greater than it would have been had an algebraic definition been at hand.

The assumed driving force persists for 0.01 second, during which time it agitates the circuit, and the disturbance thus produced persists even after the driving force is removed. Thus, although the curves of Fig. 5 are extended to the time \( t = 0.015 \), which is 0.005 second after the removal of the driving force, the disturbance has not greatly abated during this time. To extend the solution even farther requires only persistence, but the additional labour was not warranted by the uses to which the solution was to be put.

To carry out the solution as far as it is shown required 25 approximations. In order to avoid undue confusion only the first four of these are shown. The time consumed was about twelve hours. Whether this amount of labour is prohibitive or not depends upon the economic importance of the problem. It would be a comparatively unimportant industrial problem, however, that did not justify two days' work.

It is difficult to determine the accuracy of the result, owing to the indefinite nature of the driving force. In a rough way, however, some confirmation can be obtained by the argument which follows: In any two-mesh circuit such as this, the free oscillation which takes place after the driving force is removed consists of the sum of two damped sinusoids. In the present case, one of these is very pronounced, particularly in the \( x'' \) and \( x' \) curves. The other, which is of longer period, can be detected in the \( y \)'s, where it has the appearance of a slow drift. As the frequencies of these free oscillations are independent of the nature of the driving force and can be algebraically computed, they afford a check upon the work.

Upon applying this check, it is found that the frequencies should be 33.8 and 227 cycles, the first affected by the damping factor \( e^{-31t} \), and the second by the factor \( e^{-90} \). Computation from the various arches of the \( x'' \) curve indicates that the one frequency should lie between 225 and 231 cycles per second, as it does. The other constants are unobtainable.

Although this check is a very rough one, it is accurate enough to cause the detection of errors of the grosser sort, leaving only the possibility of minor inaccuracies, the probable magnitudes of which may be judged from those revealed by the other examples.
Example IV. Illustrating the Procedure When the Boundary Values Constitute a Singularity of the f-Functions

6. In Examples I and II, which deal with Bessel’s Equation (6), the boundary conditions are applied at \( x = 1 \). This is an ordinary point of the coefficients \( \frac{1}{x^2} - 1 \) and \( \frac{1}{x} \). If, however, the attempt is made to apply the boundary values at \( x = 0 \), which is a singular point for these coefficients, the right-hand side of (5) becomes indeterminate, and the value of \( f' \) cannot be obtained. Similar difficulties may present themselves in a variety of guises, but almost all of them can be overcome by some sort of artifice. Frequently the best plan is to determine in advance by algebraic means a set of functions, \( y_1, y_1', \ldots \), to which the true solution is known to be asymptotic in the vicinity of the troublesome point, using this set as the first approximation in the Picard process. When this can be done, the difficulty usually vanishes, and the computation can be completed according to the standard routine.

In the case of equation (6), if the boundary values* are

\[ y' = 1, \quad y = 0, \]

at \( x = 0 \), the three terms

\[ \frac{y'}{x}, \quad y, \quad \frac{y}{x^2} \]

take the limiting forms

\[ \frac{1}{0}, \quad 0, \quad \frac{0}{0} . \]

The last term is indeterminate and may either dominate the first or be negligible with respect to it, but the second term is obviously negligible by comparison with the first. Hence, in the immediate vicinity of \( x = 0 \), (6) may be replaced by

\[ \frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2} = 0. \]

This equation, however, is of a well-known type, the general solution of which is†

\[ y = ax + \frac{b}{x}. \]

*If the boundary values on \( y \) and \( y' \) were taken the same as in Examples I and II, only the value of \( x \) being changed, no solution would exist. For the general solution, \( y = A J_1(x) + B Y_1(x) \), may either vanish or become infinite in the neighbourhood of \( x = 0 \), but cannot take a finite value. The attempt to apply these boundary values would therefore fail, not because of any deficiency in the method of solution, but because it is asked to do impossible things. The boundary values used above are so chosen that a solution is possible.

†Note that this asymptotic solution, like the general solution of (6), cannot take a finite value different from zero at \( x = 0 \).
If the boundary values are to be satisfied the constants $a$ and $b$ must be taken as 1 and 0, respectively. Hence

$$y_1 = x, \quad y_1' = 1,$$

are approximate solutions in the neighbourhood of $x = 0$. Accepting these as first approximations, and substituting them in (5), $f_2'$ is found to be

$$\left( \frac{1}{x^2} - 1 \right) x - \frac{1}{x},$$

which reduces to $x$, and is no longer troublesome.

The curves $(y_1, y_1'$ and $f_2'$) are shown in Fig. 6. When $f_2'$ is twice integrated, the second set of approximations, $y_2'$ and $y_2$, is obtained, and from these the remaining approximations are deduced by the usual cycle of operations.

To show the accuracy of the final results, they are displayed in Table III, along with the true values as computed from the known solution

$$y = 2J_1(x).$$

The maximum error is about one-fiftieth of an inch on the scale of the original drawing, six approximations being sufficient to give this result.
TABLE III

COMPARISON OF RESULTS OF THE GRAPHICAL AND THE ALGEBRAIC SOLUTION OF EXAMPLE IV

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<th>y (calculated)</th>
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EXAMPLE V. ILLUSTRATING THE PROCEDURE WHEN THE BOUNDARY VALUES ARE NOT GIVEN FOR THE SAME VALUE OF x

7. The classical proof of the convergence of the Picard approximations requires that the boundary values be given in the form (2). But there are many other types of conditions which serve to define a solution equally well. For instance,

\[ \begin{align*}
    y &= \eta, \quad \text{at } x = \xi_1, \\
    y' &= \eta', \quad \text{at } x = \xi_1, \\
    y^{(n-1)} &= \eta^{(n-1)}, \quad \text{at } x = \xi_{n-1},
\end{align*} \]

is such a set, and the graphical solution can frequently be applied to cases of this sort. As an illustration, Bessel's Equation (6), may again be considered, subject to the conditions

\[ \begin{align*}
    y &= 0, \quad \text{at } x = 3, \\
    y' &= 1, \quad \text{at } x = 1.
\end{align*} \]

As in Examples I and II, the constants

\[ \begin{align*}
    y_1 &= 0, \\
    y'_1 &= 1
\end{align*} \]

are chosen as the first approximations and substituted in (5), just as if both boundary values were stated for the same value of x. The result is a second derivative curve,

\[ f' = -\frac{1}{x}, \]

which is plotted for integration (see Fig. 7). The first integral of this function is obtained in the usual way, but in tracing the second the pen of the integrator is set on the point (3, 0), and the machine is run backward to x = 1, thus assuring an integral which will take the proper boundary value. The curves thus obtained are accepted as y' and y. They are then substituted in the right-hand side of (5), a new second derivative curve \( f'' \) is plotted, and the cycle repeated as often as is required to obtain the desired degree of accuracy. In the case of Fig. 7 nine approximations were required.
The solution of the equation being

\[ y = -1.848 J_1(x) + 1.141 Y_1(x), \]

it is possible to determine the accuracy of the graphical results. The check is shown in Table IV. The maximum error, on the scale of the original drawing is about one-fiftieth of an inch.

<table>
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<tr>
<th>( x )</th>
<th>( y ) (graphical)</th>
<th>( y ) (computed)</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>-2.160</td>
<td>-2.155</td>
<td>+0.005</td>
</tr>
<tr>
<td>1.2</td>
<td>-1.971</td>
<td>-1.968</td>
<td>+0.003</td>
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<tr>
<td>1.4</td>
<td>-1.794</td>
<td>-1.788</td>
<td>+0.006</td>
</tr>
<tr>
<td>1.6</td>
<td>-1.606</td>
<td>-1.601</td>
<td>+0.005</td>
</tr>
<tr>
<td>1.8</td>
<td>-1.404</td>
<td>-1.398</td>
<td>+0.006</td>
</tr>
<tr>
<td>2.0</td>
<td>-1.186</td>
<td>-1.181</td>
<td>+0.005</td>
</tr>
<tr>
<td>2.2</td>
<td>-0.954</td>
<td>-0.951</td>
<td>+0.003</td>
</tr>
<tr>
<td>2.4</td>
<td>-0.714</td>
<td>-0.712</td>
<td>+0.002</td>
</tr>
<tr>
<td>2.6</td>
<td>-0.471</td>
<td>-0.470</td>
<td>+0.001</td>
</tr>
<tr>
<td>2.8</td>
<td>-0.229</td>
<td>-0.231</td>
<td>-0.002</td>
</tr>
<tr>
<td>3.0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

One point worthy of notice in connection with this type of boundary condition is, that it is not possible to reduce the labour by dividing the entire interval into convenient sub-intervals. Instead, every approximation must be carried through the entire interval, no matter how obviously incorrect it may be over the major part of the range. It is only by so doing that the boundary values can be fitted into the graphical scheme at all.
Example VI. Illustrating the Procedure When One Derivative is Without a Boundary Value

8. In the important special case where the equations (1) are equivalent to a single $n$'th order equation, the curve representing $y$ is frequently required to pass through $n$ preassigned points. In such cases, the boundary conditions take the form:

$$
\begin{align*}
    y &= \eta_1, & \text{at } x &= \xi_1, \\
    y &= \eta_2, & \text{at } x &= \xi_2, \\
    & \cdots \\
    y &= \eta_{(n-1)}, & \text{at } x &= \xi_{(n-1)}.
\end{align*}
$$

With such boundary values it is not feasible to start the process of approximation by assuming constant values for $y, y', \ldots, y^{(n-1)}$; because in the first place, the constant value assigned to $y$ cannot be so chosen as to satisfy more than one of the $n$ conditions imposed upon it, and because in the second place, nothing at all is known about the various derivatives, and there is therefore no rational means of assigning appropriate constants to them. Hence it is necessary to add to the usual cycle of operations some additional process by means of which the boundary values may be chosen.

Suppose, for the sake of argument, that the $(k-1)$st set of approximations has been derived, and that it has been used to obtain a new $k$'th approximation for the $n$'th derivative. The usual cycle of operations would require this $n$'th derivative to be repeatedly integrated, the results being accepted as the $k$'th set of approximations. But as the value of $y^{(n-1)}$ is not known for any value of $x$, it is impossible to determine in advance the constant of integration which should be used in performing the first integration—which means mechanically that the point from which the integraph pen should start is unknown. Under the circumstances, the most that can be done is to choose a constant at random in the hope that the mistake maybe corrected later on. The integral $\int y^{(n-1)} dx$ thus obtained, and the integral $\int y^{(n-1)} dx$ determined by the boundary values, can differ from each other only to the extent of an additive constant. Hence they must satisfy the relation

$$
\int y^{(n-1)} dx = \int y^{(n-1)} dx + C_{(n-1)}.
$$

The derivative of next lower order—the $(n-2)$nd—should be obtained by integrating $\int y^{(n-1)} dx$, but this is impossible mechanically since $y^{(n-1)}$ is not known. However, it follows from (10) that

$$
y^{(n-2)} = C^{(n-2)} + \int y^{(n-1)} dx
$$

in which the integral of $\int y^{(n-1)} dx$, like $y^{(n-1)}$ itself, is indefinite, and can be produced by the integraph. That is, no matter what constants of integration are assigned in the two integrations, the curve drawn by the integraph and the curve required by the boundary conditions can differ by a polynomial of the first degree, at most.
By continuing this argument, the relation

\[ y_k = C_k + C_k' x + \ldots + C_k^{(n-1)} \frac{x^{n-1}}{(n-1)!} + \tilde{y}_k \]  

(11)

can ultimately be deduced, the \( \tilde{y}_k \) representing any \( n \)-fold integral that might be obtained with the integraph, and \( y_k \) the particular \( n \)-fold integral which would pass through the required points.

Call the values of \( \tilde{y}_k \) at the points \( \xi_1, \xi_2, \ldots, \xi_n \), as read from the curve drawn by the integraph, \( \eta_1, \eta_2, \ldots, \eta_n \). The values of \( y_k \) at these same points are known to be \( \eta_1, \eta_2, \ldots, \eta_n \). Substituting them in (11), pair by pair, the set of \( n \) linear equations

\[ C_k + C_k' \xi_1 + C_k'' \frac{\xi_1^2}{2!} + \ldots + C_k^{(n-1)} \frac{\xi_1^{n-1}}{(n-1)!} = \eta_1 - \eta_1, \]

(12)

\[ C_k + C_k' \xi_2 + C_k'' \frac{\xi_2^2}{2!} + \ldots + C_k^{(n-1)} \frac{\xi_2^{n-1}}{(n-1)!} = \eta_2 - \eta_2, \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

\[ C_k + C_k' \xi_n + C_k'' \frac{\xi_n^2}{2!} + \ldots + C_k^{(n-1)} \frac{\xi_n^{n-1}}{(n-1)!} = \eta_n - \eta_n, \]

is derived, from which the \( n \) unknown constants \( C_k, C_k', \ldots, C_k^{(n-1)} \) can all be determined by the usual methods of solving linear equations.

Once \( C_k^{(n-1)} \) is known, it can be added to \( \tilde{y}_k^{(n-1)} \), and it will then give \( y_k^{(n-1)} \) in accordance with (10). Similarly, by adding \( C_k^{(n-2)} + C_k^{(n-1)} x \) to \( \tilde{y}_k^{(n-2)} \) the function \( y_k^{(n-2)} \) may be obtained. Proceeding in this manner, the entire set of \( k \)'th approximations may be deduced from the indefinite integrals given by the integraph.

With this extension the cycle of operations appears to be satisfactory, in the sense that a routine exists by means of which each set of approximations can be used to obtain the next set in order. It is still necessary, however, to overcome the difficulty, to which attention has already been directed, of choosing the first set in the sequence. This is most conveniently done by deriving the curve of \( (n-1) \)st degree which passes through the \( n \) given points, and accepting it, together with its successive derivatives, as the first set of approximations.

If the differential equation is of high order, this method of solution is too tedious to be practical. But most of the differential equations of applied mathematics are of the second order, and few indeed are higher than the fourth. For such equations the computations are not at all difficult.

Take, for example, Bessel's Equation (6). To solve it subject to the boundary values

\[ y = 1, \quad \text{at } x = 1, \]

\[ y = 0, \quad \text{at } x = 2, \]
a first approximation passing through these points must be obtained, and in accordance with the rule laid down above, this must be a curve of degree \( n - 1 \). As \( n = 2 \), this curve reduces to a straight line, the equation of which is obviously

\[
y_1 = 2 - x.
\]

This \( y_1 \) together with its derivative

\[
y_1' = -1
\]

constitutes the first set of approximations.

When they are substituted in (5), they lead to a second derivative

\[
f_2' = 2 \left( \frac{1}{x^3} - 1 \right) + x,
\]

which is plotted in Fig. 8. The integraph is then used to integrate this curve, giving \( y_2' \) and \( y_2 \). In the first of these the constant of integration was so chosen as to make \( y_2' = -0.6 \) when \( x = 1 \); the sole reason for this choice being that the curve occupied a convenient place on the paper. In the second integration, however, the constant was so chosen as to satisfy one of the two boundary values.

The polynomial of the degree \( n - 1 \) which converts \( y_2 \) into \( y_3 \) being of the form

\[
C + C'x,
\]

and \( \eta_1 \) having been chosen as unity, the equations (12) reduce to the simple form

\[
C + C' = 0,
\]

\[
C + 2C' = -\eta_2.
\]
The solution of these equations being
\[ C' = -\eta_2, \]
\[ C = +\eta_1, \]
it follows that
\[ y_2 = \bar{y}_2 + (1 - x) \eta_2. \]
From the figure, the value of \( \eta_2 \) is read of as 0.681, whence
\[ (13) \quad y_2 = \bar{y}_2 + 0.681(1 - x). \]
Similarly
\[ (14) \quad y_2' = \bar{y}_2' - 0.681. \]

Next, the values of \( \bar{y}_2 \) and \( \bar{y}_2' \) corresponding to a suitable set of values of \( x \) must be read from the curves. They are then corrected by means of equations (13) and (14), so as to give \( y_2 \) and \( y_2' \). These are then substituted back in (5), yielding \( f_2' \), from which in turn \( \bar{y}_2' \) and \( \bar{y}_2 \) are obtained by integration, corrected to give \( y_2' \) and \( y_2 \), and so on.

As the curve is a simple one the process converges so rapidly that only four approximations are required. As a check upon the accuracy of the results obtained, they are recorded in Table V, together with the values computed from the known solution
\[ y = 0.2652J_1(x) + 0.9101K_1(x). \]
The maximum error, on the scale of the original drawing, was about one two-hundredth of an inch, the high degree of accuracy being a direct result of the rapidity of convergence.

### Table V

**Comparison of Results of the Graphical and the Algebraic Solution of Example VI**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y ) (graphical)</th>
<th>( y ) (calculated)</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1.000</td>
<td>1.000</td>
<td>0.000</td>
</tr>
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<td>0.646</td>
<td>0.000</td>
</tr>
<tr>
<td>1.4</td>
<td>0.541</td>
<td>0.541</td>
<td>0.000</td>
</tr>
<tr>
<td>1.5</td>
<td>0.441</td>
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<td>0.081</td>
<td>0.000</td>
</tr>
<tr>
<td>2.0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>
9. The general method of solution to which attention has been called above is virtually identical with Picard's method of approximation. So long as the boundary values are given in the form (2), the departures from the Picard process are of negligible consequence, and the usual proof of convergence, with appropriate minor changes, is still valid.

When the form of the boundary values is altered, however, somewhat more extensive changes are required in the Picard process, and the convergence of the sequence of functions is no longer assured. Examples V and VI show that the process sometimes converges, and if it does, it can only converge to the true solution. But it does not follow that it always converges. As a matter of fact, it is not difficult to adduce cases in which the opposite is true.

When such cases occur, it is still possible to fall back upon the expedient of using a cut-and-try process. For instance, if a differential equation were to be solved subject to the boundary values

\[ y = 0, \text{ at } x = 0, \]
\[ y' = 1, \text{ at } x = 3, \]

and if the process explained in §7 failed, the equation might be solved subject to several different sets of conditions such as

\[ y = 0, \quad y = 0, \quad y = 0, \]
\[ y = 0, \quad y' = 1, \quad y' = 2, \]

all applied at \( x = 0 \). From the final curve for each of these cases the value of \( y' \) at \( x = 3 \) could then be read off, after which by interpolation, a new value of \( y' \) could be so chosen at \( x = 0 \) as to cause the value of \( y' \) at \( x = 3 \) to be very close to unity, as desired.

For instance, if when \( y' \) is given the values 0, 1 and 2 at \( x = 0 \), it is found to have the values 0.6, 0.9, 1.1 at \( x = 3 \), interpolation suggests that the value \( y' = 1.5 \) at \( x = 0 \) ought to lead to \( y' = 1.0 \) at \( x = 3 \). On trial, the latter value might be found to be 1.02, and then a further interpolation would be resorted to, giving the value 1.42 at \( x = 0 \) as a plausible one to try. By patience, the solution would ultimately be obtained in this way. In general, however, the labour would be prohibitive, and it would be necessary to seek some other method of solution.

In spite of such failings, however, the Picard method, in combination with the integraph, has a wide field of usefulness in the solution of such problems as are met in industrial research. Not the least of its advantages lies in the fact that no extensive knowledge of mathematics is required to carry out the cycle of operations. This makes it possible to turn the actual work of solving the problems over to competent computers, such as are available in most industrial institutions, thus releasing the engineer for other activities.
In conclusion, some mention should be made of several other closely related discussions of graphical and mechanical methods for solving differential equations. The one which bears the closest resemblance to the present paper is contained in the book, *Graphical Methods*, by Carl Runge. Two methods are there presented, one founded upon Picard's approximation scheme, the other upon the method of Cauchy-Lipschitz. Both contemplate the use of purely graphical integration—which is not a satisfactory process in general—but can be equally well treated by means of the integraph. When so regarded, it is identical with the process explained in Example I.

Ernesto Pascal, in a booklet entitled *I miei Integrafi per Equazioni Differenziale*, deals with the problem of designing machines to integrate various types of equations, explaining in detail a large number of such machines. Where the work of a particular industry is such that equations of a certain type arise with great frequency, the possession of a machine especially designed for this type of equation may be advisable. In other establishments it is desirable to have one machine which is universally useful. No existing machine fully satisfies this requirement as to versatility, but the integraph approaches it most nearly.

In addition to these books, which contain a large proportion of original work, the excellent general reviews given by Maurice D'Ocagne in the second volume of his *Cours de Géométrie* and by Runge in the Enzyklopädie der Mathematischen Wissenschaften will be found helpful by those who are interested in the subject of graphical solution of differential equations*.

*Since the above was written (Aug. 7, 1924), Professor V. Bush of the Massachusetts Institute of Technology has developed a mechanical device of remarkable versatility in the solution of differential equations. See, for example, the advance press notices as quoted in The Literary Digest for Dec. 17, 1927.