THE TOPOLOGICAL INVARIANTS OF ALGEBRAIC VARIETIES

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1. Since the origins of the birational geometry of algebraic varieties can be traced back to Riemann's theory of algebraic functions, it is not surprising that topological considerations have played a considerable role in the theory of algebraic varieties defined over the field of complex numbers. For instance in the theory of algebraic functions of two complex variables, due mainly to Picard, topological considerations are used extensively, and in the more geometrical theory of surfaces, due to the Italian school of geometers, many arguments are based essentially on topological methods. But in all this work the topology is subservient to function-theoretic or geometrical ends, and when Lefschetz's memoir [9] of 1921 appeared, followed, in 1924, by his Borel Tract, L'analyse situe et la geometrie algebrique, a new chapter in the theory of algebraic varieties was opened in which the topological properties of varieties are given equal status with the geometrical and function-theoretic properties. The researches thus initiated have now taken their place in the general theory of complex manifolds and have made important contributions to the problem of classifying these manifolds. In this lecture, I propose to give some account of the present state of our knowledge of the topological properties of algebraic varieties and of the contributions of this theory to the theory of complex manifolds.

What I have to say about algebraic varieties is valid, strictly speaking, only for algebraic varieties in complex projective space which are without singular points. A few of the results may be generalised to apply to varieties with singular points of a sufficiently simple character, but in what follows we shall not consider such generalisations, and we shall henceforth assume that all the varieties considered are non-singular.

2. The investigations described in Lefschetz's Borel Tract fall into two classes in the first, there is a straightforward study of an algebraic variety \( V_m \), of \( n \) dimensions, as a topological space, and in the second, the variety is investigated as a complex differentiable manifold. In the investigation of the purely topological properties of \( V_m \), the fact that there exist systems of algebraic subvarieties of \( V_m \) is fundamental, and the whole of Lefschetz's investigation depends on the selection of a suitable system. Broadly speaking, any system \( | V_{m-1} | \) of varieties on \( V_m \) which correspond to the hyperplane sections of a variety in bi-regular birational correspondence with \( V_m \) will serve, though this is, in fact, a more severe restriction than is necessary. We shall refer to the system selected as the selected "allowable system". By means of an allowable system \( | V_{m-1} | \) it is possible to determine the topological properties of \( V_m \) from a knowledge of the geometrical and topological properties of \( | V_{m-1} | \). In this way, Lefschetz is able to establish \( \}

1 Numbers in brackets refer to the references at the end of the paper.
number of general theorems on the topology of algebraic varieties. Not only is it possible to compute the homology groups of $V_m$ and to establish such general theorems as

(a) the Betti numbers $R_p$ of even dimension of an algebraic variety are always positive;

(b) the Betti numbers of odd dimension are always even;

(c) for $p \leq m$, $R_p \geq R_{p-2}$;

but it is possible to classify the $p$-cycles of $V_m$ into subsets in a significant way. This classification is not an absolute classification, but is relative to the selected allowable system. We denote by $V_r$ the intersection of $m - r$ general varieties $V_{m-1}$ of the selected allowable system, or the Riemannian manifold of $2r$ real dimensions corresponding to it. The results obtained by Lefschetz, or directly reducible from these, are as follows.

I. If $p \leq m$ and $\Gamma_p$ is any $q$-cycle of $V_m$ $(q \leq p)$, there is a $q$-cycle in $V_p$ homologous to $\Gamma_p$.

II. Every homology between the $q$-cycles of $V_m$ lying in $V_p$ $(p > q)$ which holds in $V_m$ also holds in $V_p$.

III. If $p \leq m$, a basis for the $p$-cycles of $V_m$ can be chosen as follows:

1. $R_p - R_{p-2}$ independent $p$-cycles lying in $V_p$, no combination of which is homologous to a cycle in $V_{p-1}$. Cycles homologous to any combination of these $\gamma$-cycles are called effective $p$-cycles;

2. $R_{p-2} - R_{p-4}$ independent $p$-cycles lying in $V_{p-1}$, no combination of which is homologous to a cycle in $V_{p-2}$. Cycles homologous to any combination of these $\gamma$-cycles are said to be ineffective of rank 1;

3. $q = \lfloor p/2 \rfloor$, $R_{p-2q}$ independent $p$-cycles lying in $V_{p-q}$. Cycles homologous to any combination of these $p$-cycles are said to be ineffective of rank $q$.

IV. If $\Gamma_p$ is an ineffective $p$-cycle of rank $q$, the intersection $\Gamma_p \cdot V_{m-r}$ is an ineffective $(p - 2r)$-cycle of rank $q - r$ if $r \leq q$, and is homologous to zero if $r > q$.

V. Corresponding to each effective $p$-cycle $\Gamma_p$ there is a $(2m - p)$-cycle $\Gamma_{2m-p}$ whose homology class is uniquely defined, with the property $\Gamma_{2m-p} \cdot V_p \sim \Gamma_p$. These dual cycles have the property that if $\Gamma_{2m-p+2r}$ is the dual of an effective $(p - 2r)$-cycle, $\Gamma_{2m-p+2r} \cdot V_{m-r}$ is an ineffective $p$-cycle of rank $r$. Further, if $\Gamma_{2m-p+2r}$ and $\Gamma_{2m-p+2s}$ are the duals of effective $(p - 2r)$- and $(p - 2s)$-cycles, $\Gamma_{2m-p+2r} \cdot \Gamma_{2m-p+2s} \cdot V_{m-r-s}$ is homologous to zero if $r \neq s$.

We shall return to these results later. We now consider the properties of $V_m$ as a differentiable manifold. In order to describe Lefschetz's contribution to this theory, we recall that any unmixed algebraic subvariety of dimension $d$ of $V_m$ defines a cycle on $V_m$ of dimensions $2d$. The main results of Lefschetz which I quote refer to the case $m = 2$, that is, algebraic surfaces. The first tells us that if $U_1$ and $U_1'$ are two curves on the surface $V_2$, a necessary and sufficient condition
that they belong to the same algebraic system of curves on \( V \) is that they be homologous as 2-cycles; this extends a result due to Severi [12] which proves that if \( U_1 \) and \( U'_1 \) are homologous as cycles there must exist a non-zero integer \( \lambda \) such that \( \lambda U_1 \) and \( \lambda U'_1 \) are algebraically equivalent. Using another result due to Severi [12], it is possible to show that there exists a finite subset of 2-cycles, every one of which corresponds to a curve on \( V \), which has the property that any curve on the surface is algebraically equivalent to some integral combination of the set—in fact, we get the theory of the base for surfaces. The generalisation of this to subvarieties of dimension \( d \) of a variety of \( m \)-dimensions is not yet complete in the case \( d < m - 1 \).

The second result of Lefschetz tells us that a necessary and sufficient condition that a 2-cycle in \( V \) be algebraic, that is, homologous to a cycle defined by a curve on \( V \), is that the algebraic double integrals of the first kind attached to \( V \) should all have period zero on \( \Gamma \). This result has many geometrical applications; as an example, the theorem, taken with the theory of the base, enables us to develop very easily the Hurwitz theory of correspondences between algebraic curves [10].

It is clearly a matter of great importance to extend Lefschetz’s condition for a 2-cycle to be algebraic. The general problem is as follows. It follows from the general topological properties of an algebraic variety, described above, that a \( p \)-cycle on \( V \) is always homologous to a cycle lying in an algebraic subvariety \( U \) of \( V \), where \( q \leq p \). If it is homologous to a cycle lying in some subvariety \( U \), we say that it is of rank \( k \). Clearly \( k \leq \lfloor p/2 \rfloor \). If \( p \) is even and \( k = p/2 \), the cycle is homologous to some \( U \), that is, it is algebraic. A necessary condition that a \( p \)-cycle \( \Gamma \) be of rank \( k \) is

\[
\int_{\Gamma} A^{(p-k, h)} = 0,
\]

for every exact \( p \)-form which can be written in the form

\[
A^{(p-k, h)} = \sum_{1 \leq \alpha_1 \leq 2} \ldots \sum_{1 \leq \alpha_p \leq 2} A_{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_k} \prod_{i=1}^{p-k} dz^{\alpha_i} \prod_{i=p-k+1}^{p} dz^{\beta_i},
\]

which is finite everywhere on \( V \), and for which \( h < k \). Lefschetz’s theorem tells us that this condition is sufficient for the case \( m = p = 2 \), and another of his results shows that it is sufficient for the case \( p = 2m - 2 \), any \( m \). Subsequently, I [7] was able to show that if the condition is satisfied for \( p = 2 \), any \( m \), there exists a non-zero integer \( \lambda \) such that \( \lambda \Gamma \) is algebraic, and recently [8] I have proved the sufficiency of the condition in the case \( m = p = 3 \) when the hyperplane sections of \( V \) have geometric genus equal to zero. Beyond this, the problem is an unsolved one; I believe, however, that if we can solve it for the case \( p = m \) (any \( m \)) and \( k = 1 \), a general solution may be deduced.

3. These results due to Lefschetz were the starting point of an extensive investigation of an algebraic variety as a complex differentiable manifold. A
complex manifold of one dimension is always the Riemann surface of an algebraic curve, uniquely defined to within a birational transformation, but a similar result does not hold for complex manifolds of more than one dimension; there are complex differentiable manifolds of dimension $m$ ($m > 1$) which are not algebraic. Let us, for the moment, turn our attention to a general complex differentiable manifold $M_m$ of $m$ dimensions. On this, we can construct multiple integrals, and for these integrals we have the theorems of de Rham [3]: if $\Gamma_{2m-p}$ is any $(2m-p)$-cycle on $M_m$, there exists an exterior $p$-form $A$ with coefficients which are functions of $z_1, \ldots, z_m, \bar{z}_1, \ldots, \bar{z}_m$ regular in the domain of the parameters $z_1, \ldots, z_m$, which is exact, and which satisfies the condition

$$\int_{\Gamma_p} A = I(\Gamma_{2m-p}, \Gamma_p)$$

for every $p$-cycle $\Gamma_p$, where $I(\Gamma_{2m-p}, \Gamma_p)$ is the intersection number of $\Gamma_{2m-p}$ and $\Gamma_p$. For brevity, we say that $A$ is homologous to $\Gamma_{2m-p}$, writing $A \sim \Gamma_{2m-p}$. Conversely, given any exact $p$-form $A$ which is regular on $M_m$, there exists a $(2m-p)$-cycle $\Gamma_{2m-p}$, defined to within its homology class, which is homologous to $A$, the cycles being considered as elements of a vector space over the field of complex numbers. $\Gamma_{2m-p} \sim 0$ is a necessary and sufficient condition that $A$ should be the derived form $dB$ of a $(p-1)$-form $B$.

It is always possible to attach a Hermitian metric of class $r$ to a complex manifold. A complex manifold to which a Hermitian metric is attached is called a Hermitian manifold. Topologically, it has the same degree of generality as the general complex manifold, but the existence of the metric allows us to introduce two new concepts, harmonic integrals and characteristic classes.

An exterior $p$-form $A$ is harmonic if it has the following property. We define the derivative of $A$ in a $p$-fold direction at a point $O$ as

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{E_\delta} A,$$

where $E_\delta$ is a $p$-cell containing $O$, of measure $\delta$, which is tangent to the given direction at $O$, and whose diameter tends to zero with $\delta$. It is easy to show that there exists a unique $(2m-p)$-fold form $*A$ whose derivative at every point $O$ in any $(2m-p)$-fold direction is equal to the derivative of $A$ in the absolutely perpendicular direction. $A$ is harmonic if it is exact and if the dual form $*A$ is also exact; $*A$ is then also harmonic. The fundamental theorem is that there is one and only one harmonic form $A$ homologous to any given $(2m-p)$-cycle $\Gamma_{2m-p}$.

The characteristic classes on a Hermitian manifold $H_m$ are defined by Chern [2] as follows. Let $K_{2m}$ be complex covering $H_m$, and let $K_s$ denote its $s$-dimensional skeleton. Consider the skeleton $K_{2m-2r+1}$. It is possible to define, in a continuous manner, an ordered set of $r$ mutually orthogonal unit complex vectors in $H_m$ at all points of $K_{2m-2r+1}$. Let $E_{2m-2r+2}$ be any $(2m-2r+2)$-cell
of \( K_{2m} \). Its boundary \( \partial E_{2m-2r+2} \) is a \((2m - 2r + 1)\)-sphere, and the vectors given on \( K_{2m-2r+1} \) map this boundary on the space \( U_{m,r} \) of all ordered sets of \( r \) mutually orthogonal unit vectors of \( m \)-dimensional complex vector space. Now the homotopy groups of \( U_{m,r} \) of dimension less than \( 2m - 2r + 1 \) are all zero, while the homotopy group of dimension \( 2m - 2r + 1 \) is an infinite cyclic group. We choose a generator \( \gamma \) of this group once and for all. If \( \partial E_{2m-2r+2} \) is mapped on \( a\gamma \), we attach the integer \( a \) to \( E_{2m-2r+2} \). The \((2m - 2r + 2)\)-cells of \( K_{2m} \), with coefficients attached in this way, form a cocycle, and the cohomology class to which this cocycle belongs is independent of the field of \( r \) vectors given on \( K_{2m-2r+1} \), and also of the Hermitian metric chosen. It is called the \( r \)th characteristic class of \( H_m \), and is an invariant of the complex manifold. A \((2r - 2)\)-cycle dual to the cocycles of the \( r \)th characteristic class is called a characteristic cycle; it is sometimes more convenient to deal with characteristic cycles than with characteristic cohomology classes.

4. In order that an algebraic variety can be treated as a Hermitian manifold we have to specify a metric on it. While this can be done arbitrarily, it is desirable that the metric chosen should have some intrinsic relationship with the structure of the variety. Such a metric can be found in this way. Let \( V_{m-1} \) be any algebraic variety belonging to an allowable system on \( V_m \). Then it can be shown that there exists an exact 2-form

\[
\omega = \sum a_{\alpha \beta} dz^\alpha dz^\beta
\]

homologous to \((-1)^{1/2} \times V_{m-1}\), such that the quadratic differential form

\[
\sum a_{\alpha \beta}(dz^\alpha dz^\beta)
\]

defines a positive definite Hermitian metric on \( V_m \). We use this metric, which bears a special relationship to the selected allowable system \( |V_{m-1}| \).

The chosen metric has the property

\[
\frac{\partial a_{\alpha \gamma}}{\partial z^\beta} = \frac{\partial a_{\beta \gamma}}{\partial z^\alpha}, \quad \frac{\partial a_{\alpha \beta}}{\partial \bar{z}^\gamma} = \frac{\partial a_{\gamma \beta}}{\partial \bar{z}^\alpha},
\]

which is equivalent to the condition that the fundamental 2-form \( \omega \) associated with the metric is exact. Such a metric is called a Kähler metric. Kähler metrics appear in so many places in mathematics that it is worth while studying Kähler manifolds, that is, Hermitian manifolds with Kähler metrics, in some detail. On a general Kähler manifold \( \mathbb{K}_m \) the \((2m - 2)\)-cycle \( \Delta \) homologous to the fundamental 2-form \( \omega \) is not restricted to be a multiple of a cycle whose cells have integral coefficients, as is the case when we consider algebraic varieties.

Not all complex manifolds carry Kähler metrics. A property possessed by all Kähler manifolds, but not by more general complex manifolds, is easily established. The forms

\[
\omega = \omega_1, \quad \omega_2 = \omega_1 \times \omega, \cdots, \quad \omega_r = \omega_{r-1} \times \omega
\]
are all exact. If $\omega_s$ were derived, so would $\omega_r$ be, for $s > r$. But $\omega_m = m!$ times the element of volume of $\mathcal{K}_m$, and cannot be a derived form since its integral over $\mathcal{K}_m$ is not zero. Hence no $\omega_s$ is derived, and the $2(m - r)$-cycle homologous to $\omega_r$ is not bounding, for $r = 1, \cdots, m$. It follows that the Betti numbers $R_{2n}$ of even dimension are positive for $\mathcal{K}_m$.

5. Kähler manifolds have, however, many other special properties. Some of these have recently been described by Eckmann and Guggenheim [6]. The properties of the metric implied by the Kähler condition lead to a classification of the harmonic integrals into subclasses. It can be shown that, if $p \leq m$, there are exactly $R_p - R_{p-2}$ independent $p$-forms $P$ which are harmonic and satisfy the condition $P \times \omega_{m-p+1} = 0$. The forms satisfying these conditions are called the effective $p$-forms. An important fact concerning an effective $p$-form is that the equation $P \times \omega_m = 0$ implies that $P$ is zero.

If $p \geq 2$, there are $R_{p-2} - R_{p-4}$ independent effective $(p - 2)$-forms; the product of any effective $(p - 2)$-form by $\omega_2$ is a harmonic $p$-form, which we say is ineffective of rank one. Similarly, if $p \geq 4$, we have $R_{p-4} - R_{p-6}$ independent effective $(p - 4)$-forms, and the product of an effective $(p - 4)$-form by $\omega_2$ is a harmonic $p$-form, ineffective of rank 2. And so on. The $(R_p - R_{p-2}) + (R_{p-2} + R_{p-4}) + \cdots = R_p$ harmonic forms so constructed are linearly independent, and form a base for the set of harmonic $p$-forms.

We call attention to the similarity between this classification of harmonic forms and Lefschetz's classification of cycles, described earlier. This similarity is not accidental. Let $\Delta$ be a $(2m - 2)$-cycle homologous to the fundamental form $\omega$, and let $\Gamma_{2m-p+2r}$ be a cycle homologous to an effective $(p - 2r)$-form $P$. Since $P$ is effective, $P \times \omega_{m-p+r}$ is a non-zero harmonic $(2m - p)$-form. Also [3]

$$P \times \omega_{m-p+r} \sim \Gamma_{2m-p+2r} \cdot (\Delta)^{m-p+r},$$

hence $\Gamma_{2m-p+2r} \cdot (\Delta)^{m-p+r}$ is a $p$-cycle which is not homologous to zero. Corresponding to the $R_{p-2r} - R_{p-2r-2}$ ineffective $p$-forms $P \times \omega_r$ of rank $r$, we obtain $R_{p-2r} - R_{p-2r-2}$ $p$-cycles $\Gamma_{2m-p+2r} \cdot (\Delta)^{m-p+r}$, which we call ineffective cycles of rank $r$. In the case of an algebraic variety, in which $\Delta$ is, save for a factor, a cycle representing a variety of the selected allowable system, it is easy to show that these ineffective $p$-cycles of rank $r$ are just the ineffective $p$-cycles of rank $r$ of the Lefschetz classification, and most of the results obtained by Lefschetz can then be read off from the corresponding results for the harmonic forms. Thus nearly all of the results of Lefschetz follow directly from the fact that an algebraic variety carries a Kähler metric associated with the selected allowable system. It can further be shown that the ineffective $p$-forms of rank $r$ have zero periods on the ineffective $p$-cycles of rank $r$, and from this it follows that the essential properties of harmonic $p$-forms can easily be deduced once one knows the properties of the effective $p$-forms, and it is necessary to study only the latter. This applies to all values of $p$ not exceeding $m$; the theory for values of $p$ exceeding $m$ is easily deduced by duality.
The effective $p$-forms on a Kähler manifold can be further resolved. We call a $p$-form of the type
\[
\sum_{a_1, \ldots, a_h=1}^m \sum_{\beta_1, \ldots, \beta_k=1}^m A_{a_1 \ldots a_h \beta_1 \ldots \beta_k} dz^{a_1} \cdots dz^{a_h} dz^{\beta_1} \cdots dz^{\beta_k}
\]
($h + k = p$) a form of type $(h, k)$. Any $p$-form $A$ can be written uniquely as a sum of $p$-forms, one of type $(p, 0)$, one of type $(p - 1, 1)$, \ldots, one of type $(0, p)$. If $A$ is effective, it is a consequence of the Kählerian condition that each of these forms is effective. We can resolve the vector space of dimension $R_p - R_{p-2}$ of effective $p$-forms into the sum of a space of dimension $\sigma^{p,0}$ formed by the effective forms of type $(p, 0)$, a space of dimension $\sigma^{p-1,1}$ formed by the effective forms of type $(p - 1, 1)$, and so on. If $A$ is an effective form of type $(h, k)$ homologous to the $(2m - p)$-cycle $\Gamma_{2m-p}$, its complex conjugate $\tilde{A}$ is an effective form of type $(k, h)$, homologous to the complex conjugate cycle $\tilde{\Gamma}_{2m-p}$. In particular, $\sigma^{h,k} = \sigma^{k,h}$. From this it follows that, if $p$ is odd, $R_p$ is even. The numbers $\sigma^{h,k}$, for all $h, k$ such that $h + k \leq m$, can be shown to be invariants of the differential manifold which carries the Kähler metric. In the case of an algebraic variety, $\sigma^{p,0}$ is the number of linearly independent exact algebraic $p$-fold integrals of the first kind attached to the variety, an important invariant in the classical theory of algebraic varieties.

This further subdivision of the effective $p$-forms leads at once to a further classification of the effective $p$-cycles. If $\Gamma_{i,h,k}$ ($i = 1, \ldots, \sigma^{h,k}$) are the $(2m - p)$-cycles corresponding to a basis for the effective $p$-forms of type $(h, k)$, it can be shown that the intersection matrix
\[
|| I(\Gamma_{i,h,k} \cdot \Gamma_{i',h',k'}^\dagger \cdot (\Delta)^{m-p}) ||
\]
($h + k = p = h' + k'$) is zero unless $h = h', k = k'$, and is, save for a scalar multiplier, a positive definite Hermitian matrix when $h = h', k = k'$. These properties have no counterpart in the Lefschetz theory. However, if we impose on our Kähler manifold a further condition, that the $(2m - 2)$-cycle $\Delta$ be a scalar multiple of a cycle $\Gamma$ whose cells have integral coefficients, we are able to obtain new theorems about the “integral” topology of $\mathcal{X}_m$. This condition is fulfilled for algebraic varieties.

When this new condition is satisfied, it is easy to show that the classification of the $p$-cycles of $\mathcal{X}_m$ into effective cycles, etc., can be carried out rationally, that is, that a basis for the ineffective $p$-cycles of rank $r$ can be found consisting of $R_{p-2r} - R_{p-2r-2}$ integral cycles ($r = 0, 1, 2, \ldots$). The further classification, corresponding to the classification of forms according to their type $(h, k)$, is not rational, but from the results stated above for this classification we can deduce that if $I$ is the intersection matrix of effective $p$-cycles (regarded as cycles in $(\Gamma)^{m-p}$), $I$ is symmetric when $p$ is even and has the signature $^3

^3$ If $M$ is any real symmetric matrix, we can find a real non-singular matrix $T$ such that $TMT^\dagger$ has $+1$ in the first $\alpha$ places on the principal diagonal, $-1$ in the next $\beta$ places on the principal diagonal, and zero everywhere else. Whatever matrix $T$ is used to reduce $M$ to this diagonal form, we always obtain the same numbers $\alpha, \beta$. $(\alpha, \beta)$ is called the signature of $M$ [4].
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\[(\sigma^{p,0} + \sigma^{p-2,3} + \ldots + \sigma^{0,p} + \sigma^{p-1,1} + \sigma^{p-3,3} + \ldots + \sigma^{1,p-1}).\]

In the case \(p = 2\), this gives \(\sigma^{2,0}\) as a pure topological invariant of \(\mathcal{X}_m\). Another result of considerable interest is that when \(\Delta = k\Gamma\), \(\Gamma\) being an integral cycle, the period matrix of the \(\sigma^{1,0}\) independent everywhere finite simple integrals \(\int A_i ds^4\) is a Riemann matrix, and hence, if \(R_1 > 0\), it is possible, by the use of \(\Theta\)-functions, to define one-valued functions on \(\mathcal{X}_m\) which are algebroid at all points.

6. The results which we have described show that many of the classical invariants of an algebraic variety \(V_m\) appear as invariants of the complex manifold associated with it, and that some are pure topological invariants. Let us consider the case \(m = 2\), that is, of an algebraic surface. The number \(\sigma^{1,0} = R_1/2\) is equal to the number of linearly independent simple algebraic integrals of the first kind attached to \(V_2\), and this is known to be equal to the irregularity. Hence we have

\[R_1 = 2(p_g - p_n).\]

Stated in a slightly different way, another result described above tells us that if the intersection matrix \(I\) of the 2-cycles of \(V_2\) is reduced to diagonal form by a transformation \(\text{TTT}'\), the number of positive terms on the diagonal is \(2\sigma^{2,0} + 1 = 2p_g + 1\). Thus both \(p_g\) and \(p_n\) can be determined as pure topological characters. It has also been shown by Alexander [1] that the Euler-Poincaré invariant of \(V_2\) is equal to \(I + 4\), when \(I\) is the Zeuthen-Segre invariant of the surface. Since the linear genus \(v\) of the surface is connected with \(I\) and \(p_n\) by the formula

\[I + v = 12p_n + 9,\]

we conclude that the four principal numerical invariants of a surface can all be expressed in topological terms.

7. So far, we have dealt only with topological properties of a variety which can be deduced by means of the theory of harmonic integrals. Let us now consider the characteristic classes of the Kähler manifold defined by an algebraic variety \(V_m\) and a selected allowable system on it. The determination of the dual characteristic cycles for an algebraic variety \(V_m\), in the case in which \(V_m\) carries a sufficient number of algebraically independent simple integrals of the first kind, can be deduced from results of Severi [13], Todd [14], and Eger [5]. For, any simple integral \(\int A_i ds^4\) defines a covariant vector-field \(A_i\), and when we have \(r\) of these which are algebraically independent, we can use them to define a field of \(r\) independent vectors on the skeleton \(K_{2m-2r+1}\) of a covering complex \(K_{3m}\). From these, by a standard process, we can deduce an ordered set of \(r\) mutually orthogonal unit vectors. It is then not difficult to show that to define the coefficient associated with any \((2m - 2r + 2)\)-cell of the covering complex in the corresponding characteristic cocycle we have only to take the intersection number of this cell with the locus, of \(r - 1\) complex dimensions, of points of \(V_m\) at which
the Jacobian matrix of the simple integrals has rank less than \( r \). It then follows that the cycles of the \( r \)th characteristic homology class are homologous to the variety of points of \( V_m \) at which the Jacobian of the simple integrals has rank less than \( r \). This locus has been determined, in the case \( m = 2, r = 1 \), by Severi; in the case \( r = 1 \) (any \( m \)) by Todd; and generally by Eger. It is a variety of the canonical system \( X_{r-1}(V_m) \), as defined by Severi [13], Segre [11], Todd [15, 16], and Eger [5].

The cycles of the \( r \)th characteristic homology class, as determined above differ in sign from those defined by Chern [2], since our computation has made use of covariant vectors, while Chern uses contravariant vectors. According to Chern’s definition, the cycles of the \( r \)th characteristic class are homologous to \((-1)^{m-r+1} X_{r-1}(V_m)\).

This determination of the characteristic classes on an algebraic variety is valid, however, only in the case of a variety carrying a sufficiently large number of algebraically independent simple integrals. Nor does Todd’s geometrical theory of canonical systems lead directly to a proof of the desired result in the general case, and to prove it we have to proceed along other lines.

Let \( \psi \) be a rational function on \( V_m \), \( U \) its locus of zeros, and \( U' \) its locus of poles. We construct a covering complex \( K_{2m} \) so that \( U \) is covered by a subcomplex of \( K_{2m} \), and \( U' \) by a subcomplex of the dual complex. The vector \( \frac{\partial \psi}{\partial z} = \psi_i \) is defined on all cells of \( K_{2m} \) except those which meet \( U \), but it is possible to modify \( \psi_i \) in the neighbourhood of \( U' \) so that we have a vector \( \xi_i \) defined on the skeleton \( K_{2m-2r+1} \). Let \( \xi_i^{(\alpha)} (\alpha = 1, \ldots, r) \) be \( r \) vectors defined on \( K_{2m-2r+1} \), independent of \( \xi_i \). We can use the vectors \( \xi_i, \xi_i^{(1)}, \ldots, \xi_i^{(r)} \) to determine a cocycle of the \((r + 1)\)th characteristic class on \( V_m \), and the orthogonal projections of \( \xi_i^{(1)}, \ldots, \xi_i^{(r)} \) on \( U \) to determine a cocycle of the \( r \)th characteristic class on \( U \).

It is then possible to determine a relation between these. If we define characteristic classes by means of covariant vectors, and denote by \( X_{r-1}(V_m) \) a cycle of the \( r \)th characteristic homology class on \( V_m \), we obtain the following homology:

\[
X_{r-1}(U) \sim [X_r(U) + X_r(V_m)] \cdot U.
\]

Now Chern has shown that, for a projective space \( S_m \), \( X_{r-1}(S_m) \) is homologous to a cycle representing a variety of the \((r - 1)\)-dimensional canonical system. From this, the adjunction formula just obtained enables us to establish the identity of the characteristic cycles and canonical cycles on any algebraic variety.

This result raises a number of other questions which remain to be investigated. For instance, it is known that the stationary points of an effective integral of type \((m, 0)\) form a variety of the \((m - 1)\)-dimensional canonical system, but what do we know about the stationary points of integrals of type \((h, k)\)? I mention only one preliminary result which I have found: If \( m = 2 \), the stationary points of an effective integral of type \((1, 1)\) form a locus of two real dimensions, in general, which is a cycle of the second characteristic class.
8. While it would be easy to draw rash conclusions, it is surely not without significance that so many of the basic concepts of classical algebraic geometry have a counterpart in the theory of complex manifolds, and the subject calls for much further investigation. I conclude with some remarks of a very general nature.

The results which I have described fall into two classes. In the one, we use only the fact that an algebraic variety is a complex manifold; although we make use of a Hermitian metric on it, this is only incidental, for a Hermitian metric can be attached to any complex variety, and the results to which I refer—for example, those dealing with characteristic classes—do not depend on our choice of metric. In the other, the fact that the manifold carries a Kähler metric is fundamental, and we have seen that this imposes a considerable restriction on the topology of the manifold. Yet many of the results do not depend on the actual Kähler metric selected, while others depend on the fact that the fundamental 2-form \( \omega \) is homologous to a scalar multiple of an integral cycle, but beyond this do not depend on the choice of metric. We are therefore led to ask whether it is possible to characterise, in any reasonable way, the complex manifolds which can carry a Kähler metric, and in particular those on which the Kähler metric can be chosen so that the associated 2-form \( \omega \) is homologous to a multiple of an integral cycle. It is known that not all Kähler manifolds are algebraic; for instance, the generalised complex torus can be made into a Kähler manifold by a suitable choice of metric, but it is not algebraic unless its period matrix is Riemannian. On the other hand, I know of no example of a Kähler manifold whose fundamental 2-form \( \omega \) is homologous to a multiple of an integral cycle, except the algebraic manifolds. There is surely a problem of great interest and importance here, which is a challenge to all interested in the theory of complex manifolds.

References

3. G. de Rham, Journal de Mathématiques (9) vol. 10 (1931) p. 115.
11. B. Segre, Memoire della Accademia d'Italia vol. 5 (1934) p. 479.

The proofs of new results referred to in this lecture will be published in two papers in Proc. London Math. Soc. (3) vol. 1 (1951).

**GENERAL REFERENCES**


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