BASIC IDEAS OF A GENERAL THEORY OF STATISTICAL DECISION RULES

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1. Introduction. The theory of statistics has had an extraordinarily rapid growth during the last 30 years. The line of development has been largely set by two great schools, the school of R. A. Fisher and that of Neyman and Pearson. A basic feature of the theories represented by these schools is the development of various criteria for the best possible use of the observations for purposes of statistical test and estimation procedures. In this connection I would like to mention the basic notions of efficiency and sufficiency introduced by Fisher, and that of the power of a test introduced by Neyman and Pearson. It is unnecessary to dwell on the importance of these notions, since they are well known to all statisticians.

Until about 10 years ago, the available statistical theories, except for a few scattered results, were restricted in two important respects: (1) experimentation was assumed to consist of a single stage, i.e., the number of observations to be made was assumed to be fixed in advance of experimentation; (2) the decision problems treated were restricted to special types known in the literature under the names of testing a hypothesis, point and interval estimation. In the last few years a general decision theory has been developed (see, for example, [1]) that is free of both of these restrictions. It allows for multi-stage experimentation and it includes the general multi-decision problem.

I would like to outline the principles of this general theory and some of the results that have been obtained.

A statistical decision problem is defined with reference to a sequence $X = \{X_i\}$ ($i = 1, 2, \ldots, \text{ad inf.}$) of random variables. For any sequence $x = \{x_i\}$ ($i = 1, 2, \ldots, \text{ad inf.}$) of real values, let $F(x)$ denote the probability that $X_i < x_i$ holds for all positive integral values $i$. The function $F(x)$ is called the distribution function of $X$. A characteristic feature of any statistical decision problem is that $F$ is unknown. It is merely assumed to be known that $F$ is a member of a given class $\Omega$ of distribution functions. The class $\Omega$ is to be regarded as a datum of the decision problem. Another datum of the decision problem is a space $D$, called decision space, whose elements $d$ represent the possible decisions that can be made by the statistician in the problem under consideration.

For the sake of simplicity, we shall assume for the purposes of the present discussion that (1) each element $F$ of $\Omega$ is absolutely continuous, i.e., it admits a probability density function; (2) the space $D$ consists of a finite number of elements, $d_1, \ldots, d_k$ (say); (3) experimentation is carried out sequentially, i.e., the first stage of the experiment consists of observing the value of $X_1$. After the value of $X_1$ has been observed, the statistician may decide either to terminate

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1 Numbers in brackets refer to the references at the end of the paper.
experimentation with some final decision \( d \), or to observe the value of \( X_2 \). In
the latter case, after \( X_1 \) and \( X_2 \) have been observed, the statistician may again
decide to terminate experimentation with some final decision \( d \), or to observe
the value of \( X_3 \), and so on.

The above assumptions on the spaces \( \Omega \) and \( D \) and the method of experimenta-
tion are replaced by considerably weaker ones in the general theory given in
[1]. These assumptions are made here merely for the purpose of simplifying
the discussion.

A decision rule \( \delta \), that is, a rule for carrying out experimentation and making
a final decision \( d \), can be given in terms of a sequence of real-valued and Borel
measurable functions \( \delta_{im}(x_1, \ldots, x_m) \) (\( i = 0, 1, \ldots, k; m = 0, 1, 2, \ldots \), ad inf.)
where \( x_1, x_2, \ldots \), etc. are real variables and the functions \( \delta_{im} \) are subject to
the following two conditions:

\[
\delta_{im} \geq 0; \quad \sum_{i=0}^{k} \delta_{im} = 1 \quad (i = 0, 1, \ldots, k; m = 0, 1, 2, \ldots \), ad inf.).
\]

The decision rule \( \delta \) is defined in terms of the function \( \delta_{im} \) as follows: Let
\( x_i \) denote the observed value of \( X_i \). At each stage of the experiment (after
the \( m \)th observation, for each integral value \( m \)) we compute the values
\( \delta_{0m}(x_1, \ldots, x_m), \delta_{1m}(x_1, \ldots, x_m), \ldots, \delta_{km}(x_1, \ldots, x_m) \) and then perform an
independent random experiment with the possible outcomes \( 0, 1, 2, \ldots, k \) con-
structed so that the probability of the outcome \( i \) is \( \delta_{im} \). If the outcome is a
number \( i > 0 \), we terminate experimentation with the decision \( d_i \). If the out-
come is 0, we make an additional observation (we observe the value of \( X_{m+1} \))
and repeat the process with the newly computed values \( \delta_{0,m+1}, \delta_{1,m+1}, \ldots, \delta_{k,m+1}, \) and so on.

The above described decision rule may be called a randomized decision rule,
since at each stage of the experiment an independent chance mechanism is used
to decide whether experimentation be terminated with some final decision \( d \) or
whether an additional observation be made. The special case when the functions
\( \delta_{im} \) can take only the values 0 and 1 is of particular interest, since in this case
the decision to be made at each stage of the experiment is based entirely on the
observed values obtained and one can dispense with the use of an independent
chance mechanism. We shall call a decision rule \( \delta = \{ \delta_{im} \} \) nonrandomized if
the functions \( \delta_{im} \) can take only the values 0 and 1. The question whether it is
sufficient to consider only nonrandomized decision rules for the purposes of
statistical decision making is of considerable interest. We shall return to this
question later.

2. Loss, cost, and risk functions. A basic problem in statistical decision theory
is the problem of a proper choice of a decision rule \( \delta \). In order to judge the rela-
tive merits of the various possible decision rules, it is necessary to state the cost
of experimentation and the relative degree of preference we would have for the
various possible final decisions \( d \) if some element \( F \) of \( \Omega \) were known to us to be
he true distribution. The latter may be described by a non-negative function \( W(F, d) \), called loss function, which expresses the loss suffered by the statistician when the decision \( d \) is made and \( F \) happens to be the true distribution of \( X \). In most decision problems each element \( d \) of \( D \) can be interpreted as the decision to accept the hypothesis that the unknown distribution \( F \) is an element of a given subclass \( \omega_i \) of \( \Omega \). In such a case we put \( W(F, d_i) = 0 \) when \( F \in \omega_i \) and \( >0 \) when \( F \notin \omega_i \). The cost of experimentation can be described by a sequence of non-negative functions where \( c_m(x_1, \cdots, x_m) \) denotes the cost of experimentation if the experiment consists of \( m \) observations and \( x_i \) is the observed value of \( X_i \) \((i = 1, \cdots, m)\). The loss and cost functions are to be regarded as data of the decision problem. The cost function \( c_m(x_1, \cdots, x_m) \) is, of course, assumed to be Borel measurable.

Let \( p(m, d_i \mid \delta, x_1, \cdots, x_m) \) denote the conditional probability that experimentation will consist of \( m \) observations and the decision \( d_i \) will be made when \( \delta \) is the decision rule adopted and \( x_j \) is the observed value of \( X_j \) \((j = 1, \cdots, m)\).

2.1) \[ p(m, d_i \mid \delta, x_1, \cdots, x_m) = \delta_{00} \delta_{01}(x_1) \cdots \delta_{0, m-1}(x_1, \cdots, x_{m-1}) \delta_{im}(x_1, \cdots, x_m). \]

For any positive integral value \( m \), let \( f(x_1, \cdots, x_m \mid F) \) denote the joint density function of \( X_1, \cdots, X_m \) when \( F \) is the true distribution of \( X \). The expected loss, i.e., the expected value of \( W(F, d) \) depends only on the true distribution \( F \) and the decision rule \( \delta \) adopted. It is given by the expression

\[ n(F, \delta) = \sum_{m=1}^{\infty} \sum_{i=1}^{k} W(F, d_i) \delta_{0i} + \sum_{m=1}^{\infty} \sum_{i=1}^{k} \int_{R_m} W(F, d_i) p(m, d_i \mid \delta, x_1, \cdots, x_m) f(x_1, \cdots, x_m \mid F) \, dx_1, \cdots, dx_m. \]

Here \( R_m \) denotes the space of all \( m \)-tuples \((x_1, \cdots, x_m)\).

The expected cost of experimentation depends only on the true distribution \( F \) and the decision rule \( \delta \) adopted. It is given by

2.3) \[ r_2(F, \delta) = \sum_{m=1}^{\infty} \sum_{i=1}^{k} \int_{R_m} c_m(x_1, \cdots, x_m) p(m, d_i \mid \delta, x_1, \cdots, x_m) f(x_1, \cdots, x_m \mid F) \, dx_1, \cdots, dx_m. \]

Let

2.4) \[ r(F, \delta) = r_1(F, \delta) + r_2(F, \delta). \]

The quantity \( r(F, \delta) \) is called the risk when \( F \) is true and the decision rule \( \delta \) adopted. For any fixed decision rule \( \delta^0 \), the risk is a function of \( F \) only. We shall call \( r(F, \delta^0) \) the risk function associated with the decision rule \( \delta^0 \).

It is perhaps not unreasonable to judge the merit of any particular decision rule entirely on the basis of the risk function associated with it. We shall say
that the decision rule $\delta^1$ is uniformly better than the decision rule $\delta^2$ if $r(F, \delta^1) \leq r(F, \delta^2)$ for all $F$ and $r(F, \delta^1) < r(F, \delta^2)$ for at least one member $F$ of $\Omega$. A decision rule $\delta$ will be said to be admissible if there exists no uniformly better decision rule. Two decision rules $\delta^1$ and $\delta^2$ will be said to be equivalent if they have identical risk functions, i.e., if $r(F, \delta^1) = r(F, \delta^2)$ for all $F$ in $\Omega$. For any $\epsilon > 0$, two decision rules $\delta^1$ and $\delta^2$ will be said to be $\epsilon$-equivalent if $|r(F, \delta^1) - r(F, \delta^2)| \leq \epsilon$ for all $F$ in $\Omega$.

3. Elimination of randomization when $\Omega$ is finite. It was proved recently by Dvoretzky, Wolfowitz, and the author [2] that if $\Omega$ is finite, then for every decision rule $\delta$ there exists an equivalent nonrandomized decision rule $\delta^*$. Thus, in this case one can dispense with randomization and it is sufficient to consider only nonrandomized decision rules. A similar result for a somewhat more special type of randomized decision rule than the one described here was obtained independently by Blackwell [3]. The proof is based on an extension of a theorem by Liapounoff [4] concerning the range of a vector measure. The continuity of the distribution of $X_1$ (implied by our assumption of absolute continuity of $F$) is essential for the above stated result. In case of discontinuous distributions there may exist randomized decision rules with risk functions having some desirable properties that cannot be achieved by any nonrandomized decision rule.

The finiteness of the space $\Omega$ is a very restrictive condition which is seldom fulfilled in statistical decision problems. However, it was shown in [2] that for any decision rule $\delta$ and for any $\epsilon > 0$ there exists an $\epsilon$-equivalent and nonrandomized decision rule $\delta^*$ under very general conditions which are usually fulfilled in decision problems arising in applications.

An interesting result on the possible elimination of randomization, but of a somewhat different nature, was found recently by Hodges and Lehmann [5]. They proved that if the decision problem is a point estimation problem, if $D$ is a Euclidean space, and if the loss $W(F, d)$ is a convex function of $d$ for every $F$, then for any randomized decision rule $\delta$ (with bounded risk function) there exists a nonrandomized decision rule $\delta^*$ such that $r(F, \delta^*) \leq r(F, \delta)$ for all $F$ in $\Omega$. It is remarkable that neither the finiteness of $\Omega$ nor the continuity of its elements $F$ are needed for the validity of this result.

4. A convergence definition in the space of decision rules and some continuity theorems. A natural convergence definition in the space of decision rules would seem to be the following one: $\lim_{i \to \infty} \delta^i = \delta^0$ if $\lim_{i \to \infty} p(m, d_i | \delta^i, x_1, \ldots, x_m) = p(m, d_i | \delta^0, x_1, \ldots, x_m)$ for all $m$, all $i > 0$, and for all $x_1, \ldots, x_m$. This convergence definition is, however, too strong for our purposes. Instead, we shall adopt the following weaker one: We shall say that

$$\lim_{i \to \infty} \delta^i = \delta^0$$

if...
(4.2) \[ \lim_{j \to \infty} \delta_j^i = \delta_0^i \quad (i = 0, 1, \cdots, k) \]

and

\[ \lim_{j \to \infty} \int_{S_m} p(m, d_i | \delta_j^i, x_1, \cdots, x_m) \, dx_1 \cdots dx_m = \int_{S_m} p(m, d_i | \delta_0^i, x_1, \cdots, x_m) \, dx_1 \cdots dx_m \]

(4.3) \hspace{1cm} (i = 0, 1, \cdots, k; m = 1, 2, \cdots, \text{ad inf.})

holds for every measurable subset \( S_m \) of the space of all \( m \)-tuples \((x_1, \cdots, x_m)\).

It was shown (see, for example, Theorem 3.1 in [1]) that adopting the above convergence definition the following theorem holds.

**Theorem 4.1.** The space of all decision rules is compact, i.e., every sequence \( \{\delta_j^i\} \) \((j = 1, 2, \cdots, \text{ad inf.})\) of decision rules admits a convergent subsequence.

The above theorem is a simple consequence of known theorems on the “weak” compactness of a set of functions (see, for example, Theorem 17b (p. 33) of [6]).

Before stating certain continuity theorems, we shall formulate two conditions concerning the loss and cost functions.

**Condition I.** \( W(F, d_i) \) is a bounded function of \( F \) for \( i = 1, 2, \cdots, k \).

**Condition II.** The cost function has the following properties: (i) \( c_m(x_1, \cdots, x_m) \geq 0 \); (ii) \( c_{m+1}(x_1, \cdots, x_{m+1}) \geq c_m(x_1, \cdots, x_m) \); (iii) \( c_m(x_1, \cdots, x_m) \) is a bounded function of \( x_1, \cdots, x_m \) for every fixed \( m \); (iv) \( \lim_{m \to \infty} c_m(x_1, \cdots, x_m) = \infty \) uniformly in \( x_1, \cdots, x_m \).

The following continuity theorems have been proved in [1]:

**Theorem 4.2.** Let \( \{\delta_j^i\} \) \((j = 0, 1, 2, \cdots, \text{ad inf.})\) be a sequence of decision rules such that \( \lim_{j \to \infty} \delta_j^i = \delta_0^i \) and such that

\[ \delta_{01}(x_1) \delta_{02}(x_1, x_2) \cdots \delta_{0N}(x_1, \cdots, x_N) = 0 \quad (j = 0, 1, 2, \cdots, \text{ad inf.}) \]

identically in \( x_1, \cdots, x_N \) for some positive integer \( N \). Then, if Conditions I and II hold, we have \( \lim_{j \to \infty} r(F, \delta_j^i) = r(F, \delta_0^i) \) for all \( F \).

**Theorem 4.3.** If \( \lim_{j \to \infty} \delta_j^i = \delta_0^i \) and if Conditions I and II hold, then \( \liminf_{j \to \infty} r(F, \delta_j^i) \geq r(F, \delta_0^i) \) for all \( F \).

5. **Bayes and minimax solutions of the decision problem.** In this section we shall discuss the notions of Bayes and minimax solutions and some of their properties. These solutions are not only of intrinsic interest, but they play an important role in the construction and characterization of complete classes of

\footnote{Theorem 3.1 in [1] is actually much stronger and more difficult to prove, since \( D \) is not assumed there to be finite.}
decision rules discussed in the next section. We shall start out with some definitions.

By an a priori probability distribution $\xi$ in $\Omega$ we shall mean a non-negative and countably additive set function $\xi$ defined over a properly chosen Borel field of subsets of $\Omega$ for which $\xi(\Omega) = 1$. The Borel field is chosen such that $r(F, \delta)$ is a measurable function of $F$ for every fixed $\delta$.

For any a priori probability distribution $\xi$, let

$$r^*(\xi, \delta) = \int_{\Omega} r(F, \delta) \, d\xi.$$  

A decision rule $\delta^0$ is said to be a Bayes solution relative to the a priori distribution $\xi$ if

$$r^*(\xi, \delta^0) = \min_{\delta} r^*(\xi, \delta).$$

A decision rule $\delta^0$ is said to be a Bayes solution in the strict sense if there exists an a priori distribution $\xi$ such that $\delta^0$ is a Bayes solution relative to $\xi$.

A rule $\delta^0$ is said to be a Bayes solution relative to the sequence $\{\xi_i\} (i = 1, 2, \cdots, \text{ad inf.})$ of a priori distributions if

$$\lim_{i \to \infty} [r^*(\xi_i, \delta^0) - \inf_{\delta} r^*(\xi_i, \delta)] = 0$$

where the symbol $\inf_{\delta}$ stands for infimum with respect to $\delta$.

We shall say that a decision rule $\delta^0$ is a Bayes solution in the wide sense if there exists a sequence $\{\xi_i\}$ of a priori distributions such that $\delta^0$ is a Bayes solution relative to $\{\xi_i\}$.

A decision rule $\delta^0$ is said to be a minimax solution if

$$\sup_{\delta} r(F, \delta^0) \leq \sup_{\delta} r(F, \delta) \text{ for all } F,$$

where the symbol $\sup_{\delta}$ stands for supremum with respect to $F$.

An a priori distribution $\xi_0$ is said to be least favorable if the following relation is satisfied:

$$\inf_{\delta} r^*(\xi_0, \delta) \geq \inf_{\delta} r^*(\xi, \delta) \text{ for all } \xi.$$

The reason that an a priori distribution $\xi_0$ satisfying the above relation is called least favorable is this: If an a priori distribution $\xi$ actually exists and is known to the statistician, a satisfactory solution of the decision problem is to use a Bayes solution $\delta$ relative to $\xi$ since $\delta$ minimizes the average risk (averaged in accordance with the a priori distribution $\xi$). The minimum average risk that can be achieved will generally be different for different a priori distributions and an a priori distribution $\xi$ may be regarded the less favorable from the point of view of the statistician the greater the minimum average risk associated with $\xi$. Thus, an a priori distribution satisfying (5.5) will be least favorable from the point of view of the statistician.
We shall state some of the results obtained concerning Bayes and minimax solutions.

**Theorem 5.1.** If Conditions I and II hold, for any a priori distribution $\xi$, there exists a decision rule $\delta$ such that $\delta$ is a Bayes solution relative to $\xi$.

**Theorem 5.2.** If Conditions I and II hold, there exists a minimax solution.

The above existence theorems can easily be derived from the theorems stated in §4. With the help of these theorems we can even prove the slightly stronger result that admissible Bayes and admissible minimax solutions always exist.

**Theorem 5.3.** If Conditions I and II hold, then a minimax solution is always a Bayes solution in the wide sense.

**Theorem 5.4.** If $\delta^0$ is a minimax solution and $\xi_0$ is a least favorable a priori distribution, then, if Conditions I and II hold, $\delta^0$ is a Bayes solution relative to $\xi_0$ and the set $\omega$ of all members $F$ of $\Omega$ for which $r(F, \delta^0) = \sup_{F, \delta^0} r(F, \delta^0)$ has the probability measure 1 according to $\xi_0$.

The last part of Theorem 5.4 implies that the risk function of a minimax solution has a constant value over a subset $\omega$ of $\Omega$ whose probability measure is 1 according to every least favorable a priori distribution $\xi$. In many decision problems the risk function of a minimax solution is constant over the whole space $\Omega$.

Some additional results can be stated if the validity of the following additional condition is postulated:

*Condition III.* The space $\Omega$ is compact and the loss function $W(F, d)$ is continuous in $F$ in the sense of the following convergence definition in $\Omega$: We shall say that
\[ \lim_{i \to \infty} F_i = F_0 \] if for every positive integer $m$ we have
\[ \lim_{i \to \infty} \int_{S_m} f(x_1, \cdots, x_m \mid F_i) \, dx_1 \cdots dx_m = \int_{S_m} f(x_1, \cdots, x_m \mid F_0) \, dx_1 \cdots dx_m \]
normally in all measurable subsets $S_m$ of the space of all $m$-tuples $(x_1, \cdots, x_m)$.

**Theorem 5.5.** If Conditions I, II, and III hold, a least favorable a priori distribution exists.

The proof of this theorem is based on the fact that the space of all probability measures $\xi$ on a compact space $\Omega$ is compact in the sense of the following convergence definition: $\lim_{i \to \infty} \xi_i = \xi_0$ if $\lim \xi_i(\omega) = \xi_0(\omega)$ for any open subset $\omega$ of $\Omega$ whose boundary has probability measure zero according to $\xi_0$. This result was proved in [1, Theorem 2.15, p. 50]. A closely related result was

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* For a detailed discussion and proofs, see §3.5 in [1].
obtained by Kryloff and Bogoliouboff [7]. Their convergence definition in the space of the probability measures $\xi$ is somewhat different from the one used here.

**Theorem 5.6.** If Conditions I, II, and III hold, a minimax solution is always a Bayes solution in the strict sense.

The above theorem is an immediate consequence of Theorems 5.4 and 5.5.

6. **Complete classes of decision rules.** A class $C$ of decision rules $\delta$ is said to be *complete* if for any rule $\delta$ not in $C$ there exists a rule $\delta^*$ in $C$ such that $\delta^*$ is uniformly better: We shall say that a class $C$ of decision rules is *essentially complete* if for any rule $\delta$ not in $C$ there exists a rule $\delta^*$ in $C$ such that $r(F, \delta^*) \leq r(F, \delta)$ for all $F$ in $\Omega$.

Clearly, if $C$ is a complete or at least an essentially complete class of decision rules, we can disregard all decision rules outside $C$ and the problem of choice is reduced to the problem of choosing a particular element of $C$. Thus, the construction of complete or essentially complete classes of decision rules is of great importance in any statistical decision problem.

The first result concerning complete classes of decision rules is due to Lehmann [8] who constructed such a class in a special case. Soon after the publication of Lehmann's paper, results of great generality were obtained. To state some of these results, let $\Delta$ denote the set of all decision rules $\delta$ with bounded risk functions. We shall say that a class $C$ of decision rules $\delta$ is complete, or essentially complete, relative to $\Delta$ if the corresponding condition is fulfilled for every $\delta$ in $\Delta$. Among others, the following results have been proved in [1].

**Theorem 6.1.** If Conditions I and II hold, then the class of all Bayes solutions in the wide sense is complete relative to $\Delta$.

**Theorem 6.2.** If Conditions I and II hold, then the closure of the class of all Bayes solutions in the strict sense is essentially complete relative to $\Delta$.

**Theorem 6.3.** If Conditions I, II, and III hold, then the class of all Bayes solutions in the strict sense is complete relative to $\Delta$.

To avoid any possibility of a misunderstanding, it may be pointed out that the notions of Bayes solutions and a priori distributions are used here merely as mathematical tools to express some results concerning complete classes of decision rules, and in no way is the actual existence of an a priori distribution in $\Omega$ postulated here.

7. **Relation to von Neumann's theory of games.** The statistical decision theory as outlined here, is intimately connected with von Neumann's theory of zero
sum two person games [9]. The normalized form of a zero sum two person game is given by von Neumann as follows: There are two players and there is given a bounded and real-valued function $K(u, v)$ of two variables $u$ and $v$ where $u$ may be any point of a space $U$ and $v$ may be any point of a space $V$. Player 1 chooses a point $u$ in $U$ and player 2 chooses a point $v$ in $V$, each choice being made in ignorance of the other. Player 1 then gets the amount $K(u, v)$ and player 2 the amount $-K(u, v)$.

Any statistical decision problem may be viewed as a zero sum two person game. Player 1 is the agency, say Nature, who selects an element $F$ of $\Omega$ to be the true distribution of $X$, and player 2 is the statistician who chooses a decision rule $\delta$. The outcome is then given by the risk $r(F, \delta)$ which depends on both the choice $F$ of Nature and the choice $\delta$ of the statistician. The theory of zero sum two person games was developed by von Neumann for finite spaces $U$ and $V$. In statistical decision problems, however, the number of strategies available to Nature (number of elements of $\Omega$) and the number of strategies (number of decision rules) available to the statistician are usually infinite. Many of the results in statistical decision theory were obtained by extending von Neumann’s theory to the case of infinite spaces of strategies. In particular, it has been shown in [1] that if Conditions I and II hold, the statistical decision problem, viewed as a zero sum two person game, is strictly determined in the sense of von Neumann’s theory, i.e.,

\[(7.1) \quad \sup_{\xi} \inf_{\delta} r^*(\xi, \delta) = \inf_{\delta} \sup_{\xi} r^*(\xi, \delta).\]

The above relation plays a fundamental role in the theory of zero sum two person games. In statistical decision theory, the above relation is basic in deriving the results concerning complete classes of decision rules, but otherwise it is of no particular intrinsic interest.

8. Discussion of some special cases. I would like to discuss briefly application of the general theory to a few special cases.

Suppose that $\Omega$ consists of two elements $F_1$ and $F_2$. According to $F_i$ the random variables $X_1, X_2, \ldots$, ad inf. are independently distributed with the common density function $f_i(t)$ ($i = 1, 2$). The decision space $D$ consists of two elements $d_1$ and $d_2$ where $d_i$ denotes the decision to accept the hypothesis that $F_i$ is the true distribution ($i = 1, 2$). Let the loss $W(F_i, d_j) = W_{ij} > 0$ when $i \neq j$ and $=0$ when $i = j$. The cost of experimentation is assumed to be proportional to the number of observations, i.e., $c_m(x_1, \ldots, x_m) = cm$ where $c$ denotes the cost of a single observation.

An a priori distribution is given by a set of two non-negative numbers $(\xi_1, \xi_2)$ such that $\xi_1 + \xi_2 = 1$. The quantity $\xi_i$ denotes the a priori probability that $F_i$ is true. It was shown by Wolfowitz and the author [10] that any Bayes solution must be a decision rule of the following type: Let $x_j$ denote the observed value of $X_j$ and let
We choose two constants \(a\) and \(b\) \((b < a)\) and at each stage of the experiment (after the \(m\)th observation for each integral value \(m\)) we compute the cumulative sum \(Z_m = z_1 + \cdots + z_m\). At the first time when \(b < Z_m < a\) does not hold, we terminate experimentation.\(^4\) We make the decision \(d_1\) (accept the hypothesis that \(F_1\) is true) if \(Z_m \leq b\), and the decision \(d_2\) (accept the hypothesis that \(F_2\) is true) if \(Z_m \geq a\). A decision rule of the above type is called a sequential probability ratio test.

Applying the complete class theorem to this case, we arrive at the following result: The class of all sequential probability ratio tests corresponding to all possible values of the constants \(a\) and \(b\) is a complete class. This means that if \(\delta\) is any decision rule that is not a sequential probability ratio test, then there exist two constants \(a\) and \(b\) such that the sequential probability ratio test corresponding to the constants \(a\) and \(b\) is uniformly better than \(\delta\).\(^5\)

Due to the completeness of the class of all sequential probability ratio tests, the problem of choosing a decision rule is reduced to the problem of choosing the values of the constants \(a\) and \(b\). A method for determining the constants \(a\) and \(b\) such that the resulting sequential probability ratio test is a minimax solution, or a Bayes solution relative to a given a priori distribution, is discussed by Arrow, Blackwell, and Girshick [11].

The properties of the sequential probability ratio tests have been studied rather extensively. The recently developed sequential analysis (see, for example, [12] and [13]) is centered on the sequential probability ratio test. It may be of interest to mention that the stochastic process represented by the sequential probability ratio test is identical with the one-dimensional random walk process that plays an important role in molecular physics.

We shall now consider the case when \(\Omega\) contains more than two but a finite number of elements. It will be sufficient to discuss the case when \(\Omega\) consists of 3 elements \(F_1\), \(F_2\), and \(F_3\), since the extension to any finite number \(> 3\) is straightforward. As before, the random variables \(X_1, X_2, \ldots\) are independently distributed with the common density function \(f_i(t)\) when \(F_i\) is true \((i = 1, 2, 3)\). The decision space \(D\) consists of 3 elements \(d_1, d_2,\) and \(d_3\) where \(d_i\) denotes the decision to accept the hypothesis that \(F_i\) is true. Let \(W(F_i, d_j) = W_{ij} = 0\) for \(i = j\), and \(> 0\) when \(i \neq j\). The cost of experimentation is again assumed to be proportional to the number of observations, and let \(c\) denote the cost of a single observation. Any a priori distribution \(\xi = (\xi^1, \xi^2, \xi^3)\) can be represented by a point with the coordinates \(\xi^1, \xi^2,\) and \(\xi^3\). The totality of all possible a priori distributions \(\xi\) will fill out the triangle \(T\) with the vertices \(V_1, V_2, V_3\) where \(V_i\) represents the a priori distribution whose \(i\)th component \(\xi^i\) is equal to 1 (see fig. 1).

\(^4\) If \(Z_m = a\) or \(= b\), the statistician may use any chance mechanism to decide whether to terminate experimentation or to take an additional observation.

\(^5\) This result follows also from an optimum property of the sequential probability ratio test proved in [10].
In order to construct a complete class of decision rules for this problem, it is necessary to determine the Bayes solution relative to any given a priori distribution $\xi_0 = (\xi_{10}, \xi_{20}, \xi_{30})$. Let $x_i$ denote the observed value of $X_i$. After $m$ observations have been made the a posteriori probability distribution $\xi_m = (\xi_{1m}, \xi_{2m}, \xi_{3m})$ is given by the following expression:

$$
\xi'_m = \frac{\xi_{10} f_1(x_1)f_2(x_2) \cdots f_3(x_m)}{\sum_{j=1}^{3} \xi_{10}^j f_1(x_1)f_2(x_2) \cdots f_3(x_m)}
$$

(8.2)

At each stage of the experiment, the a posteriori probability distribution $\xi_m$ is represented by a point of the triangle $T$.

It was shown by Wolfowitz and the author [14] that there exist three fixed (independent of the a priori distribution $\xi_0$), closed and convex subsets $S_1$, $S_2$, and $S_3$ of the triangle $T$ such that the Bayes solution relative to $\xi_0$ is given by the following decision rule: At each stage of the experiment (after the $m$th observation, for $m = 0, 1, 2, \ldots$) compute the point $\xi_m$ in $T$. Continue taking additional observations as long as $\xi_m$ does not lie in the union of $S_1$, $S_2$, and $S_3$. If $\xi_m$ lies in the interior of $S_i$, stop experimentation with the final decision $d_i$ ($i = 1, 2, 3$). If $\xi_m$ lies on the boundary of $S_i$, an independent chance mechanism may be used to decide whether experimentation be terminated with the final decision $d_i$ or whether an additional observation be made.

The convex sets $S_1$, $S_2$, and $S_3$ depend only on the constants $W_{ij}$ and $c$. So far no method is available for the explicit computation of the sets $S_1$, $S_2$, and $S_3$ for given values of $W_{ij}$ and $c$. The development of a method for the explicit determination of the sets $S_1$, $S_2$, and $S_3$ would be of great value, since it would probably indicate a way of dealing with similar difficulties in many other sequential decision problems.

More general results concerning the nature of the Bayes solution for the decision problem described above, admitting also nonlinear cost functions, were obtained by Arrow, Blackwell, and Girshick [11].

The boundary points of the sets $S_1$, $S_2$, and $S_3$ on the periphery of the triangle $T$ and the tangents at these points have been determined in [14].
As a last example, consider the following decision problem: It is known that $X_1, X_2, \cdots$ are independently and identically distributed and the common distribution is known to be rectangular with unit range. Thus, the midpoint $\theta$ of the range is the only unknown parameter. The common density function of the chance variables $X_1, X_2, \cdots$ is given by

$$f(t, \theta) = \begin{cases} 1 & \text{when } |t - \theta| \leq 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

For any real value $\theta^*$ let $d_{\theta^*}$ denote the decision to estimate $\theta$ by the value $\theta^*$. The decision space $D$ consists of the elements $d_{\theta^*}$ corresponding to all real values $\theta^*$. Let the loss be given by $(\theta - \theta^*)^2$ when $\theta$ is the true value of the midpoint of the range and the decision $d_{\theta^*}$ is made. The cost of experimentation is assumed to be proportional to the number of observations made. Let $c$ denote the cost of a single observation.

It was shown in [1] that a minimax solution for this problem is given by the following decision rule: Take at least one observation. At each stage of the experiment (after the $m$th observation, for each positive integral value $m$) compute the quantity

$$l_m = 1 + \min(x_1, \cdots, x_m) - \max(x_1, \cdots, x_m).$$

Continue experimentation as long as $l_m > (24c)^{1/3}$. At the first time when $l_m \leq (24c)^{1/3}$ stop experimentation and estimate $\theta$ by the value

$$\theta^* = \frac{\min(x_1, \cdots, x_m) + \max(x_1, \cdots, x_m)}{2}.$$

The risk function associated with this minimax solution is constant over the whole space $\Omega$. The admissibility of the above minimax solution was proved by C. Blyth.

9. Concluding remark. While the general decision theory has been developed to a considerable extent and many results of great generality are available, explicit solutions have been worked out so far only in a relatively small number of special cases. The mathematical difficulties in obtaining explicit solutions, particularly in the sequential case, are still great, but it is hoped that future research will lessen these difficulties and explicit solutions will be worked out in a great variety of problems.

References

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