MEASURE THEORY

ON HAUSDORFF MEASURES

Aryeh Dvoretzky

Let \( y = h(x) \) be defined for \( 0 < x < \infty \) and assume values in \( 0 \leq y \leq +\infty \). Let \( S \) be any linear set of points and \( \rho \) an arbitrary positive number. Cover \( S \) by a countable number of open intervals \( I_1, I_2, \ldots \) of lengths \( x_1, x_2, \ldots \) each of which is less than \( \rho \), and denote by \( m_\rho(S; h) \) the lower bound of \( h(x_1) + h(x_2) + \cdots \) for all such coverings of \( S \). Then \( m(S; h) = \lim_{\rho \to 0} m_\rho(S; h) \) is called the (linear, exterior) Hausdorff measure of \( S \) with respect to \( h(x) \). (F. Hausdorff [Math. Ann. vol. 79 (1918) pp. 157–179] originally considered the most important case of continuous monotone \( h(x) \) with \( \lim_{x \to 0} h(x) = 0 \). For later studies see e.g. G. Bouligand [Les définitions modernes de la dimension, Actualités scientifiques et industrielles, no. 274, Paris, 1935] and A. Dvoretzky [Proc. Cambridge Philos. Soc. vol. 44 (1948) pp. 13–16].)

If \( m(S; h) < \infty \) implies \( m(S; g) < \infty \), we write \( g < h \). If either \( g < h \) or \( h < g \) we say that \( g(x) \) and \( h(x) \) are comparable; if both relations hold we write \( g \sim h \) (read: equivalent). If \( m(S; g) = m(S; h) \) for all \( S \) we write \( g \approx h \) (read: strictly equivalent). Given any \( h(x) \) it can be shown that there exists a continuous monotone (nondecreasing) \( g(x) \) strictly equivalent to \( h(x) \).

Whatever \( h(x) \) we put \( h^*(x) = \inf_{0 < t \leq x} h(t)/t \). Then \( h^* \sim h \) or, more precisely, for all \( S \) we have \( m(S; h^*) \leq m(S; h) \leq 2m(S; h^*) \).

Using this result it can be shown that \( g < h \) if and only if

\[
\limsup_{x \downarrow 0} \frac{g^*(x)}{h^*(x)} < \infty.
\]

(Hence \( g < h \) could have been defined also by the requirement that \( m(S; g) > 0 \) imply \( m(S; h) = 0 \).) Thus, \( g(x) \) and \( h(x) \) are equivalent if and only if \( h^*(x)/g^*(x) \) is bounded and bounded away from zero as \( x \downarrow 0 \); moreover, the bounds of this expression may be employed to obtain explicit bounds for \( m(S; h)/m(S; g) \).

By constructive methods similar to those used in proving the above results it can be shown that \( \forall \lim_{x \downarrow 0} h(x) = 0 \) and \( \lim_{x \downarrow 0} h(x)/x = \infty \), then there exist a monotone, continuous and convex \( g(x) \), and perfect nowhere dense sets \( S_1, S_2, S_3 \) such that \( 0 = h(S_1) < h(S_2) < h(S_3) = \infty \) and \( 0 = g(S_1) > g(S_2) > g(S_3) = 0 \). Thus the only functions \( h(x) \) comparable to all others are either 1) those for which \( \lim \inf_{x \downarrow 0} h(x) = \alpha > 0 \), when \( m(S; h) \) is \( \alpha \) times the number of points in \( S \); or 2) those for which \( \lim \inf_{x \downarrow 0} h(x)/x = \beta < \infty \), when \( m(S; h) \) is \( \beta \) times the exterior Lebesgue measure of \( S \). Similar results may be obtained for sequences of functions \( h(x) \).

Hausdorff measures of nonlinear sets can be treated in a similar manner.

HEBREW UNIVERSITY,
JERUSALEM, ISRAEL.

477
LUSIN'S THEOREM FOR ONE TO ONE MEASURABLE TRANSFORMATIONS

CASPER GOFFMAN

A one to one transformation \( T_n : (f(x), f^{-1}(y)) \) of the closed unit \( n \) cube \( I \) onto the closed unit \( n \) cube \( I^{-1} \) is said to be measurable if \( f(x) \) and \( f^{-1}(y) \) are both measurable functions; it is said to be of Baire class \( \leq \alpha \) if \( f(x) \) and \( f^{-1}(y) \) are both of Baire class \( \leq \alpha \). It is shown, for every \( n \geq 1 \), that if \( T_n \) is measurable then, for every \( \epsilon > 0 \), there are closed sets \( E \subseteq I \) and \( E^{-1} \subseteq I^{-1} \) of measure greater than \( 1 - \epsilon \) such that \( T_n \) is a homeomorphism between \( E \) and \( E^{-1} \); and there is a \( T'_n : (g(x), g^{-1}(y)) \) of Baire class \( \leq 2 \) such that \( f(x) = g(x) \) and \( f^{-1}(y) = g^{-1}(y) \) almost everywhere. If \( n \geq 2 \) then, for every \( \epsilon > 0 \), there is a homeomorphism \( T'_n : (g(x), g^{-1}(y)) \) between \( I \) and \( I^{-1} \) such that \( f(x) = g(x) \) and \( f^{-1}(y) = g^{-1}(y) \) on sets of measure greater than \( 1 - \epsilon \). This result does not hold for \( n = 1 \). Attention is called to a result the author obtained several years ago concerning arbitrary one to one transformations (Duke Math. J. vol. 10 (1943) pp. 1-4) and to a number of unsolved problems.

UNIVERSITY OF OKLAHOMA,
NORMAN, OKLA., U. S. A.

EXTENSION OF MEASURE

OTTON M. NIKODÝM

Known measure extension devices for sets are studied on abstract Boolean lattices. Let \( (A) \) be a finitely genuine Boolean sublattice of a Boolean \( \sigma \)-lattice \( (A') \), that is, for elements \( a, b, c \) of \( (A) \) the equalities

\[
a + b = c, \quad a + b' = c; \quad a \cdot b = c, \quad a \cdot b' = c
\]

are equivalent respectively, and \( 0 = 0', \ 1 = 1' \). The primed symbols refer to \( (A') \), the unprimed to \( (A) \). The above four conditions are independent. A given finitely additive measure \( \mu(a) \geq 0 \) on \( (A) \) generates the "Lebesguean exterior measure" on \( (A') \) defined by

\[
\mu'_e(a') = \inf \sum_{n=1}^{\infty} \mu(x_n)
\]

for all coverings \( \{x_n\} \) of \( a' \), that is, \( a' \subseteq \sum_{n=1}^{\infty} x_n \). The "interior measure" is defined by \( \mu'_i(a') = \mu'_e(1') - \mu'_e(0\{a'\}) \). Let \( (L') \) be the class of all \( a' \in (A') \) for which \( \mu'_e(a') = \mu'_i(a') \). \( \mu'_e(a') \) is greater than or equal to any convex Carathéodory's measure on \( (A') \) for which \( \mu'_e(a) \leq \mu(a) \) on \( (A) \). Let \( (C') \) be the class of all \( \mu'_e\)"Carathéodory-measurable" elements of \( (A') \). Let us say that \( a', b' \) do not differ by more than \( \epsilon > 0 \) if the symmetric difference \( a' + b' \) has a covering...
\{x_n\} with \(\sum_{n=1}^{\infty} \mu(x_n) \leq \epsilon\). Define \((S')\) as the class of all \(a' \in (A')\) such that for every \(\epsilon > 0\) there exist \(a \in (A)\) not differing from \(a'\) by more than \(\epsilon\). This device was studied by the author in Académie Royale de Belgique, 1938.

It can be proved that (1) the classes \((L'), (C''), (S')\) are identical and contain \((A)\), (2) they too are \(\sigma\)-lattices, and (3) \(\mu'\) is, on them, a denumerably additive measure. It may be less than \(\mu\) on some \(a \in (A)\).

If we define on \((A')\) a "fundamental" sequence \(\{a'_n\}\) as a sequence such that \(a'_p\) does not differ from \(a'_q\) by more than \(\epsilon_{p,q} > 0\) with \(\epsilon_{p,q} \rightarrow 0\), and if we use the corresponding Cantor-Ch. Meray completing process, we obtain the same Boolean lattice \((B')\) as if we had used MacNeille's well known measure extension device (Proc. Nat. Acad. Sci. U. S. A. (1938)) upon the convex measure \(\mu'\).

(These remarks represent part of the work of the author under a cooperative contract between the Atomic Energy Commission and Kenyon College.)

Kenyon College,
Gambier, Ohio, U. S. A.
SECTION III

GEOMETRY AND TOPOLOGY