AN APPLICATION OF THE MORSE THEORY TO THE TOPOLOGY OF LIE GROUPS

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1. Introduction

In [3] Hans Samelson and I applied the Morse theory to the homology of symmetric spaces. Today I would like to report very briefly on a direct application of this theory to the stable homotopy of the classical groups[1], and to point out some unsolved questions.

2. The Freudenthal theorem for symmetric spaces

Our primary interest is in the classical compact groups; nevertheless, it is essential for our method to consider the larger family of compact symmetric spaces.

A compact homogeneous Riemannian manifold M is called symmetric if it admits an 'inverse operation', i.e. if M admits an isometry, keeping a point $P \in M$ fixed, and whose differential at P is -1.

These geometric generalizations of the compact groups seem to be the class of spaces to which the Morse theory is most applicable. The reason is that, on such a space, conjugate points have global implications:

2.1. *If P and Q are conjugate of degree k along the geodesic s, then s is contained in a k-manifold of geodesics joining P to Q.*

In general this proposition is only infinitesimally true.

The principal step towards the solution of our problem is the following Freudenthal-type theorem. Let M be a compact symmetric space and let $u = (P, Q)$ be a pair of points on M. Let $\Omega_u M$ denote the piecewise regular paths from P to Q on M, topologized as in [4], and set $S_u M$ equal to the set of geodesic segments in $\Omega_u M$. The set of geodesics of minimal length in $S_u M$ is denoted by M^u. I like to think of the step from M to M^u as an antisuspension; for instance, if M is the n-sphere S_n, and u is a pair of antipodes, then $M^u = S_{n-1}$.

Each $s \in S_u M$ has an index, $\lambda(s)$, equal to the number of conjugate points of P in the interior of s. We write $|u|$ for the *least positive number* occurring among the integers $\lambda(s)$; $s \in S_u M$. Finally, the composition $\pi_k(M^u) \to \pi_k(\Omega_u M) \to \pi_{k+1}(M)$ will be denoted by u_*.

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Theorem I. Let M be a compact symmetric space, and let $u = (P, Q)$ be a pair of points on M. Then:

2.2. *$u_* : \pi_k(M^u) \to \pi_{k+1}(M)$ is bijective for $0 < k < |u| - 1$.*

2.3. *M^u is again a symmetric space.*

By virtue of (2.3), the process of antisuspending can be iterated, and we will call a sequence of symmetric spaces

$$M_1 \to M_2 \to M_3 \to \ldots,$$

a u-sequence if each M_i is some component of M_{i+1}^u for some choice of u on M_{i+1}. For instance,

$$S_n \to S_{n+1} \to S_{n+2} \to \ldots$$

is an unending u-sequence for which Theorem I yields the usual Freudenthal suspension theorem. The following are two new examples of u-sequences:

(A) $U_n \to U_{2n}/U_n \times U_n \to U_{2n} \to \ldots,$

(B) $SO_n \to O_{2n}/O_n \times O_n \to U_{2n}/O_{2n} \to Sp_{2n}/U_{2n} \to Sp_{2n}$
$\to Sp_{4n}/Sp_{2n} \times Sp_{2n} \to U_{8n}/Sp_{4n} \to SO_{16n}/U_{8n} \to SO_{16n} \to \ldots.$

Here we have used the usual notation for the classical groups and certain of their homogenous spaces. Thus O_n denotes the group of $n \times n$ orthogonal matrices, U_n the unitary ones, and Sp_n the symplectic ones.

By a more or less explicit computation it can be shown that:

Theorem II. At each step of the sequences (A) and (B), the integer $|u|$ tends to $+\infty$ with n.

On the other hand, it is well known that π_k of each of the spaces occurring in (A) or (B) becomes independent of n for $n \gg k$. Hence Theorem I together with Theorem II yields recursion relations for these stable values of π_k. In particular one has the following corollary:

Corollary. The stable homotopy of the classical groups satisfies the relation

2.4.
$$\left.\begin{aligned} \pi_k(U) &= \pi_{k+2}(U) \\ \pi_k(O) &= \pi_{k+4}(Sp) \\ \pi_k(Sp) &= \pi_{k+4}(O) \end{aligned}\right\} \quad (k = 0, 1, 2, \ldots).$$

Here we have suppressed the index n to denote stable homotopy.

The explicit computation of these groups is now a simple matter. One obtains 0, Z; for the period of $\pi_*(U)$ and Z_2, Z_2, 0, Z, 0, 0, 0, Z, for the period of $\pi_*(O)$.

Just a word about the proof. One obtains (2.2) analogously to the usual Freudenthal theorem. By means of the Morse theory one constructs a C.W. model for $\Omega_u M$ which consists of M^u with cells of dimension greater than or equal to $|u|$ attached. This is done in two steps. First, a model is found for the subsets of $\Omega_u M$, consisting of the loops of length less than a given number a. This model is a smooth manifold with boundary. On these models a function closely related to the length function takes on its absolute minimum on a homeomorphic image of M^u. Next, by using (2.1), the effect of the other critical points can be estimated. Now the desired C.W. model for $\Omega_u M$ is obtained by applying the Morse theory on manifolds.

The second part of Theorem I is obtained by a quite elementary argument concerning the midpoint of a minimal geodesic on a symmetric space. Theorem II is a routine computation in view of the fundamental conjugacy theorems of Cartan. By virtue of these, it is sufficient to study the geodesics joining P to Q on a maximal flat torus of M, and these are surveyed rather easily.

The result (2.4) was announced in [1], where a somewhat different proof was sketched. The present point of view seems more concise, and is the one which is adopted in a forthcoming paper.

3. Remarks and problems

3.1. In a sense Theorem I states that a first approximation to $\Omega_u M$ is given by M^u. By looking at the longer geodesics in $S_u M$, one can obtain better approximations. In general the second approximation is obtained by attaching a vector-bundle, ξ, to M^u. In the case of a sphere the attaching map of ξ to M^u becomes trivial after a single suspension, and it is rather natural to ask for the proper generalization of this fact.

3.2. In [3], the additive structure of $H^*(\Omega_u M; Z_2)$ is completely determined when M is a compact symmetric space. In fact we construct a gradation preserving isomorphism of $(S_u M)^*$ onto $H^*(\Omega_u M; Z_2)$, where $(S_u M)^*$ denotes the Z_2 module generated by the points of $S_u M$ ($u = (P, Q)$ a *general* pair of points in M!) and graded by the index. On the other hand we have no general description of $H^*(\Omega_u M; Z_p)$ for p an odd prime. It also seems likely that $\Omega_u M$ has no odd torsion.

The structure of $H^*(\Omega_u M; Z)$ as a Hopf-algebra is described in [2] when M is a group. A corresponding description in general is at present not even known mod 2.

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