Integral geometry, started by the English geometer M. W. Crofton, has received recently important developments through the works of W. Blaschke, L. A. Santalò, and others. Generally speaking, its principal aim is to study the relations between the measures which can be attached to a given variety. It is my purpose in the present paper to discuss the services it can render to some problems in differential geometry.

1. Measure of the spherical image of a closed submanifold in Euclidean space

A submanifold of dimension $n$ in a Euclidean space $E^{n+N}$ of dimension $n+N$ is given by an abstract differentiable manifold $M^n$ of dimension $n$ and a differentiable map $x: M^n \rightarrow E^{n+N}$, whose Jacobian matrix has everywhere the rank $n$. We say that $M^n$ is imbedded, if the map $x$ is one-one, i.e. if $x(M^n)$ does not intersect itself. All the unit normal vectors of $x(M^n)$ form a bundle of spheres of dimension $N-1$ over $M^n$ and constitute a manifold $B_v$ of dimension $n+N-1$. If $0$ is a fixed point of $E^{n+N}$ and $S_0$ the unit hypersphere with origin $O$, there is a mapping $T: B_v \rightarrow S_0$ which maps a unit normal vector of $M^n$ into the end-point of the unit vector through $O$ and parallel to it. $T$ is a generalization of the normal mapping of Gauss in the theory of surfaces.

Suppose from now on that $M^n$ is compact. Then $B_v$ is also compact, and we define as the total curvature of $x(M^n)$ the volume of the image $T(B_v)$ divided by the volume of $S_0$ itself, each point of $T(B_v)$ counted a number of times equal to the number of points mapped into it. This total curvature we will denote by $T_x(M^n)$. It is in a sense a measure of the curvedness of the submanifold. For a closed space curve, for instance, its total curvature is, up to a constant factor, the integral of the absolute value of the curvature.

Concerning the total curvature Lashof and I proved the following theorems:

(1) The total curvature $T_x(M^n)$ is greater than or equal to the sum of the Betti numbers of $M^n$ relative to an arbitrary coefficient field. As a corollary it follows that $T_x(M^n) \geq 2$, a result which can be derived
directly by an elementary argument on the maxima and minima of the co-ordinate functions on $M^n$.

(2) If $T_x(M^n) < 3$, $M^n$ is homeomorphic to a sphere. (This result was proved also by Milnor[8].)

(3) $x(M^n)$ is a convex hypersurface imbedded in a subspace of dimension $n + 1$, if and only if $T_x(M^n) = 2$.

A basic reason for these theorems is the existence of the large number of co-ordinate functions on $M^n$. Morse's critical point theory then furnishes one of the essential tools in the proofs.

For a differentiable manifold abstractly given, one is led to study the immersions for which $T_x(M^n)$ is as small as possible. Two questions naturally arise: (a) What is the minimum value of $T_x(M^n)$, expressed in terms of $M^n$ itself only, for all possible immersions $x$? (b) To characterize the immersions $x$ for which the total curvature attains this minimum value. Theorem 3 answers these questions for the case when $M^n$ is homeomorphic to a sphere.

For general compact manifolds very little is known about the two questions. Theorem 1 implies that $T(M^n) = \min_x T_x(M^n)$ is greater than or equal to the sum of the Betti numbers mod 2 of $M^n$. There are sufficient indications to support the conjecture that $T(M^n)$ is equal to the minimum number of cells by which $M^n$ can be subdivided into a cell complex, but the truth of this remains undecided.

As for Question (b), a conjecture of N. H. Kuiper says that if $M^n$ is immersed in $E^{n+1}$ with the minimum total curvature $T(M^n)$, then $N \leq \frac{1}{2} n(n + 1)$. Some necessary conditions are known when $M^n$ is a hypersurface ($N = 1$), and has minimum total curvature.

A simple case is when $M^n$ is a closed space curve ($n = 1$, $N = 2$). Then Theorem 2 can be sharpened to the following form (Fary[6] and Milnor[8]): A closed space curve with total curvature $< 4$ is unknotted. The theorem thus gives a simple necessary condition for a knot in space.

Another application is the following consequence of the above Theorem 3: a closed surface immersed in ordinary Euclidean space with Gaussian curvature $K \geq 0$ is an imbedded convex surface. Under the stronger assumption that $K > 0$ the conclusion follows from a well-known argument of Hadamard. It may be of interest to remark that a similar statement is not valid in higher dimensions; there are examples of non-convex closed hypersurfaces in a Euclidean space of four or higher dimensions whose Gauss–Kronecker curvature is everywhere non-negative.
2. Measure of the image of a complex analytic mapping

Entirely analogous to the theory of submanifolds in Euclidean space, that of complex analytic submanifolds in a complex projective space. Let $\mathcal{M}_n$ be a complex manifold of (complex) dimension $n$ and $Z: \mathcal{M}_n \to P_{n+N}$ be a complex analytic mapping of $\mathcal{M}_n$ into the complex projective space $P_{n+N}$ of dimension $n + N$. The study of such mappings includes as particular cases various classical theories. In fact, if $\mathcal{M}_n$ is compact, $Z(\mathcal{M}_n)$ is an algebraic variety. If $\mathcal{M}_n$ is the complex Euclidean line $E_1$ (or the Gaussian plane, as it is commonly called), the complex analytic mapping $Z: E_1 \to P_1$ defines a meromorphic curve in the sense of H. Weyl, J. Weyl and Ahlfors. In particular, the notion of the complex analytic mapping $Z: E_1 \to P_1$ is identical with that of a meromorphic function defined in the Gaussian plane.

Starting with the classical theorem of Picard, a main problem in such investigations is the determination of the maximum size of the set of linear spaces of dimension $N$, which will be disjoint with the image $Z(\mathcal{M}_n)$. For meromorphic curves a satisfactory solution is provided by the following theorem of E. Borel: Suppose that the meromorphic curve is non-degenerate (i.e. that it does not lie in a hyperplane of $P_{1+N}$). Given $N + 3$ hyperplanes in general position, the image $Z(E_1)$ meets one of them. Obviously this theorem contains as a particular case the theorem of Picard that an entire function in the Gaussian plane omits at most one value.

That the theory is mainly geometrical can be justified by the following generalization of Borel’s theorem, which follows easily from results of Ahlfors: Let $Z: E_1 \to P_{1+N}$ be a non-degenerate meromorphic curve. Given $\binom{N + 2}{k + 1} + 1$ linear spaces of dimension $N - k$ in general position, $0 \leq k \leq N$, one of them must meet an osculating linear space of dimension $k$ of the curve.

In the establishment of these and related results, integral geometry plays a role on at least two occasions. Although the theorems relate only to the incidence of the curve with the linear subspaces, it is necessary to use the elliptic Hermitian metric in $P_{n+N}$. Then, for compact $\mathcal{M}_n$, $Z(\mathcal{M}_n)$ has a finite volume and this volume is, up to a numerical factor, equal to the order of the algebraic variety. This identification of volume and order, of sufficient interest in the compact case, will be of paramount importance in the case when $\mathcal{M}_n$ is non-compact. For then the notion of order does not exist, while the volume does. As it turns out, the volume does fulfil many of the functions of the order.
Since a non-compact manifold will be exhausted by a sequence of expanding polyhedra with boundaries, we are led to the study of a complex analytic mapping $Z: M_n \to P_{n+N}$, where $M_n$ is compact and is with or without boundary. The first problem is the following: Given a generic linear space $L$ of complementary dimension $N$, to determine the difference between the number of points of intersection of $L$ and $Z(M_n)$, each counted with its proper multiplicity, and the volume of $Z(M_n)$. This problem was solved by Levine\cite{7}, who expressed the difference as an integral over the boundary $\partial M_n$ of $M_n$. His result can be stated as follows:

Let $Z = (z_0, z_1, \ldots, z_{n+N}) \neq 0$ be a homogeneous co-ordinate vector of $P_{n+N}$, so that $Z$ and $\lambda Z = (\lambda z_0, \lambda z_1, \ldots, \lambda z_{n+N})$, where $\lambda$ is a non-zero complex number, define the same point. For $Z$ and $W = (w_0, w_1, \ldots, w_{n+N})$ we introduce the Hermitian scalar product

$$
(Z, W) = (\bar{W}, Z) = \sum_{k=0}^{n+N} \bar{z}_k w_k.
$$

The linear space $L$ of dimension $N$ can be defined by the equations

$$
l_i \equiv (Z, A_i) = 0 \quad (1 \leq i \leq n),
$$

where we suppose $(A_i, A_j) = \delta_{ij}$, $1 \leq i, j \leq n$. Then, for $\zeta \in M_n$, the function

$$
u(\zeta, L) = \frac{\|L\|}{|Z|} \leq 1 \quad (|Z| = + (Z_1 Z)^\frac{1}{2}, \quad \|L\| = + (l_1 l_1 + \ldots + l_n l_n)^\frac{1}{2}),
$$

where $Z = Z(\zeta)$ is a homogeneous co-ordinate vector of the image point of $\zeta$, is a well-defined real-valued function in $M_n$, and vanishes, if and only if $Z(\zeta) \in L$. Similarly, we define the exterior differential forms

$$
\Phi = \frac{i}{\pi} d'd^\ast \log \|L\|,
$$

$$
\Psi = \frac{i}{\pi} d'd^\ast \log |Z|,
$$

and

$$
\Lambda = \frac{1}{2\pi i} (d' - d^\ast) \log u \Lambda \sum_{0 \leq k \leq n-1} \Phi^k \Lambda \Psi^{n-1-k}.
$$

Then we have the formula

$$
v(M_n) - n(M_n, L) = - \int_{\partial M_n} \Lambda,
$$

where $n(M_n, L)$ is the number of points common to $L$ and $Z(M_n)$, counted with their multiplicities, and $v(M_n)$ is the volume of $Z(M_n)$, suitably normalized. It follows in particular that, if $M_n$ is without boundary and
if \( Z(M_n) \) is non-degenerate, so that \( v(M_n) \neq 0 \), then \( Z(M_n) \) meets every linear space of dimension \( N \) in \( P_{n+N} \).

Perhaps the first example of a non-compact complex manifold is the complex Euclidean space \( E^n \) of dimension \( n \). Let \( \zeta_1, \ldots, \zeta_n \) be the co-ordinates in \( E^n \). We will exhaust \( E^n \) by the domains \( M(r) \):

\[
\zeta_1 \bar{s}_1 + \cdots + \zeta_n \bar{s}_n \leq r^2,
\]

as \( r \to \infty \). This seems to be the most natural exhaustion, because if we compactify \( E^n \) by adding a hyperplane \( \pi \) at infinity, the complement of \( M(r) \) in \( E^n \) will form a tubular neighborhood about \( \pi \).

Consider first the classical case of a meromorphic function \( Z : E^n_1 \to P_1 \). Let \( v(r) \) be the volume of the image of \( M(r) \). For a generic point \( L \in P_1 \) let \( n(r, L) \) be the number of times \( L \) is covered by \( Z(M(r)) \). Then (6) can be written

\[
1 \leq 2n \leq \log v(r) - n(r, L) = -\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \log u}{\partial r} d\theta \quad (\zeta_1 = re^{i\theta}).
\]

This induces us to put

\[
T(r) = \int_{r_0}^{r} \frac{v(t)}{t} dt, \quad N(r, L) = \int_{r_0}^{r} \frac{n(t, L)}{t} dt \quad (r_0 > 0).
\]

By integrating (8) with respect to \( r \), we get

\[
T(r) - N(r, L) = -\frac{1}{2\pi} \int_0^{2\pi} \log u d\theta \bigg|_{r_0}^{r}.
\]

This is the so-called first main theorem in the theory of meromorphic functions. Our introduction of the order function \( T(r) \) is exactly the way it was introduced by Shimizu–Ahlfors.

Since the first main theorem involves a generic point \( L \) of \( P_1 \), it is natural to integrate it over \( P_1 \). If we perform the integration with the invariant density \( dL \), we shall get a formula of the Crofton type

\[
T(r) = \int_{L \in P_1} N(r, L) dL,
\]

which implies that the average of the right-hand side of (10) is zero. On the other hand, we derive from the first main theorem the fundamental inequality

\[
T(r) - N(r, L) > \text{const.}
\]

If we integrate this inequality over a non-invariant density, it is easy to get the theorem that the complement of the image set \( Z(E_1) \) in \( P_1 \) has measure zero. An idea initiated by F. Nevanlinna and simplified by Ahlfors[1] consists in the use of a density with singularities. It is the
integration of (12) relative to such a density that leads to a proof of the Picard–Borel theorem.

In the case of complex analytic mappings \( Z : E_2 \to P_2 \) there are known examples which show that the complement of the image \( Z(E_2) \) may contain open subsets of \( P_2 \). We shall give a brief discussion of the proper restrictions on the mapping \( Z \) in order that general statements can be made. In fact, the first main theorem on complex analytic functions has the following generalization:

Let \( v(r) \) be the volume of the image of \( M(r) \) and, for a generic point \( L \in P_2 \), let \( n(r, L) \) be the number of times \( L \) is covered by \( Z(M(r)) \). Let

\[
T(r) = \int_{r_0}^{r} \frac{v(t)}{t^3} \, dt, \quad N(r, L) = \int_{r_0}^{r} \frac{n(t, L)}{t^3} \, dt \quad (r_0 > 0).
\]

Then we have the inequality

\[
T(r) - N(r, L) > \text{const} - S(r, L), \tag{14}
\]

where

\[
S(r, L) = \frac{2}{\pi^2} \int_{r_0}^{r} \int \left( v_{11} + v_{22} \right) \log u \, dV \geq 0,
\]

\[
v_{kk} = \frac{\partial^2}{\partial s_k \partial s_k} \log (\| Z \cdot \| L \|) \quad (k = 1, 2), \tag{15}
\]

the integration being over the volume element \( dV \) of the unit hypersphere in \( E_n \). It is clear from this inequality that in order to have a statement on the image set \( Z(E_n) \) we must have \( T(r) \to \infty \) as \( r \to \infty \). The latter is automatic in the 1-dimensional case, but is an additional assumption in the 2-dimensional case. In fact, the well-known examples of Fatou–Bieberbach do not have this property. If, moreover,

\[
S(r, L) = o(T(r)), \tag{16}
\]

then we have the theorem that \( Z(E_2) \) omits at most a set of measure zero.

The assumption (16) is unsatisfactory in the sense that it involves the generic point \( L \in P_2 \). The expression for \( S(r, L) \) suggests that a ‘mixed order function’ should be introduced. In fact, let \( \Omega \) and \( \Omega_0 \) be the associated two-forms of \( P_2 \) and \( E_2 \) respectively. Then

\[
\int_{M(r)} Z^*(\Omega) \wedge \Omega_0 = v_1(r), \tag{17}
\]

where \( Z^*(\Omega) \) is the inverse image of \( \Omega \) under the mapping \( Z \), is a mixed volume of the domain \( M(r) \). Put

\[
S(r) = \int_{r_0}^{r} \frac{v_1(t)}{t^3} \, dt. \tag{18}
\]
It is conceivable that condition (16) can be replaced by a condition on the relative growth of \( T(r) \) and \( S(r) \).

It seems to me that these problems on complex analytic mappings deserve much further study.

\section*{Integral formulae and rigidity theorems}

I believe my discussion of relations between differential and integral geometry will leave a big gap, if I do not touch on the role that integral formulae play in the proofs of rigidity or uniqueness theorems. Perhaps the most well-known example of such considerations is Herglotz’s proof of the uniqueness of Weyl’s problem. In spite of these important applications it should be of independent interest to derive integral formulae for compact immersed submanifolds for their own sake. A little analytic manipulation shows that there are few such formulae, unless the latter are allowed to involve other geometrical elements in the space, such as fixed points, fixed linear subspaces, fixed directions, etc. The reason is simple: For an immersed submanifold \( x: M^n \to E^{n+N} \), the \( o \)-ordinate vector \( x(p), p \in M^n \), depends on the choice of the origin.

The simplest case is that of a strictly convex hypersurface \( x: M^n \to E^{n+1} \). Naturally we orient it so that the Gauss–Kronecker curvature is everywhere \( > 0 \). Since the normal mapping of the hypersurface \( \Sigma = x(M^n) \) into the unit hypersphere \( S_0 \) about the origin is one–one and has a non-zero Jacobian everywhere, the hypersurface can be defined by \( x: S_0 \to \Sigma \subset E^{n+1} \), where \( x \) maps a point \( \xi \) of \( S_0 \) into the point of \( \Sigma \) having \( \xi \) as the unit normal vector.

To get rigidity theorems suppose \( x': S_0 \to \Sigma' \) is a second strictly convex hypersurface. It is then possible to write down a number of globally defined exterior differential forms on \( S_0 \). For our purpose we shall restrict ourselves to the following:

\[
A_{rs} = (x, \xi, d\xi, \ldots, d\xi, dx, \ldots, dx, dx', \ldots, dx'), \\
A'_{rs} = (x', \xi, d\xi, \ldots, d\xi, dx, \ldots, dx, dx', \ldots, dx').
\]

Each of these expressions is a determinant of order \( n+1 \), whose rows are the components of the respective vectors or vector-valued differential forms, with the convention that in the expansion of the determinant the multiplication of differential forms is in the sense of exterior multiplication. The subscript \( r \) refers to the number of entries \( dx \) and the subscript \( s \) that of the entries \( dx' \). Since \( A_{rs} \) and \( A'_{rs} \) are globally defined on \( S_0 \), their integrals over \( S_0 \) are zero.
The integral formulae so obtained can be expressed in a more geometrical form as follows: Let \( \Pi = d\xi^2 \) be the fundamental form of \( \Sigma_0 \), and let

\[
\Pi = -dx\,d\xi, \quad \Pi' = -dx'\,d\xi
\]

be the second fundamental forms of \( \Sigma, \Sigma' \) respectively. Let \( \Delta(y, y') \) be the determinant of the ordinary quadratic differential form \( y\Pi + y'\Pi' + \Pi \) relative to a local co-ordinate system, so that \( \Delta(y, y')/\Delta(0, 0) \) is independent of the choice of the local co-ordinate system. Let

\[
\frac{\Delta(y, y')}{\Delta(0, 0)} = \sum_{0 \leq r+s \leq n} \frac{n!}{r!s!(n-r-s)!} y^r y'^s P_{rs},
\]

where \( P_{rs} \) are mixed invariants of \( \Sigma, \Sigma' \). In particular, \( P_{10}, P_{01} \) are, up to numerical factors, the \( I \)th elementary symmetric functions of the principal radii of curvature of \( \Sigma, \Sigma' \) respectively. Then our integral formulae can be written

\[
\int_{\Sigma} (pP_{rs} - P_{r+s+1}) dV = 0, \quad \int_{\Sigma} (p'P_{rs} - P_{r+s+1}) dV = 0,
\]

where \( dV \) is the volume element of \( \Sigma_0 \) and \( p, p' \) are the support functions of \( \Sigma, \Sigma' \) respectively. An important consequence of (22) consists of the formulæ

\[
\int_{\Sigma} p_{\Sigma} dV = \int_{\Sigma} p'_{\Sigma} dV, \quad \int_{\Sigma} p_{t-1,1} dV = \int_{\Sigma} p'_{t-1,1} dV \quad (l \geq 1),
\]

which give

\[
2\int_{\Sigma} (p_{\Sigma} - P_{-l+1,1}) dV = \int \{p'(P_{1,l-1} - P_{0,l}) - p(P_{l-1,1} - P_{0,l})\} dV.
\]

It is important to observe that the right-hand side of (24) is antisymmetric in the hypersurfaces \( \Sigma, \Sigma' \).

Formula (24) reduces to a purely algebraic problem the proof of the following uniqueness theorem of Minkowski, A. D. Alexandroff\footnote{Alexandroff, A. D.}, and Fenchel and Jessen\footnote{Fenchel, W.; Jessen, B.}: If two closed strictly convex hypersurfaces are such that at points with parallel normals, the \( l \)th (for fixed \( l \geq 2 \)) elementary symmetric functions of the principal radii of curvature have the same value, then they differ from each other by a translation.

The theorem is also true for \( l = 1 \), but it will have a different (and simpler) proof.

The algebraic lemma needed has been communicated to me by L. Gårding, as a consequence of his work on hyperbolic polynomials. It can be stated as follows:
Let \((\lambda_{ik})\) be an \(n \times n\) symmetric matrix, and let
\[
\det (\delta_{ik} + y\lambda_{ik}) = \sum_{0 < r < n} P_r(y), \tag{25}
\]
Let \(P_r(\lambda^{(1)}, \ldots, \lambda^{(r)})\) be the completely polarized form of \(P_r(\lambda)\), so that
\[
P_r(\lambda^{(1)}, \ldots, \lambda^{(r)}) = P_r(\lambda) \cdot r.
\]
Then, for \(r > 2\) and for positive definite matrices \((\lambda_{ik}^{(1)}), \ldots, (\lambda_{ik}^{(r)})\), the following inequality is valid:
\[
P_r(\lambda^{(1)}, \ldots, \lambda^{(r)}) \geq P_r(\lambda^{(1)})^{1/r} \cdots P_r(\lambda^{(r)})^{1/r}. \tag{26}
\]
Equality holds, if and only if the \(r\) matrices are pairwise proportional.

The uniqueness theorem then follows immediately from the lemma and the integral formula (24). For the hypothesis says that \(P_0 = P_0\).
From (26) it follows that \(P_{r-1,1} - P_0 \geq 0\). By (24) this is possible only when \(P_{r-1,1} - P_0 = 0\). Again by the lemma it follows that the second fundamental forms of the hypersurfaces are equal.

So far as I am aware, it is not known whether a similar uniqueness theorem is valid, if the \(I\)th elementary symmetric function of the principal curvatures is prescribed as a function of the normal vector. Alexandroff proved that a closed convex surface in ordinary Euclidean space is defined up to a translation, if its mean curvature is a given function of the normal. His proof made use of a maximum principle. It would be interesting if this theorem can be proved by using integral formulae.

REFERENCES
THE POLARIZATION OF ALGEBRAIC VARIETIES, AND SOME OF ITS APPLICATIONS

By T. MATSUSAKA

Let $V^m$ be a complete non-singular variety.† There are three kinds of equivalence relations of $V$-divisors, linear, algebraic and numerical equivalence, which we assume are well known. Let $G_l, G_a, G_n$ be respectively the set of $V$-divisors which are linearly, algebraically and numerically equivalent to 0. These form subgroups of the group $G$ of $V$-divisors. We say that a divisor $X$ on $V$ is linearly effective, if the complete linear system $2(X)$ determined by $X$ gives a projective embedding of $V$. Let $\mathfrak{X}$ be a non-empty set of positive $V$-divisors such that if $X$ is in $\mathfrak{X}$, a positive $V$-divisor $Y$ is in $\mathfrak{X}$ if and only if $mX = sY \mod G_n$ for a pair $(m, s)$ of positive integers. It is clear that $\mathfrak{X}$ is determined uniquely by one of the divisors contained in it. When $\mathfrak{X}$ contains at least one linearly effective divisor, then we call the pair $(V, \mathfrak{X})$ a polarized variety, polarized by the set $\mathfrak{X}$. A polarized variety $(V, \mathfrak{X})$ is an algebraic variety plus an additional structure determined by $\mathfrak{X}$. When $V$ is a curve, we can put on $V$ one and only one structure of polarization. The same is true in general if $G/G_n \simeq \mathbb{Z}$ and if $V$ admits a projective embedding. The notion of polarization was first introduced by Weil (cf. Weil\cite{9}) and studied by the present writer (cf. Matsusaka\cite{30}).

1. The group of automorphisms

Let $V, V'$ be two complete non-singular varieties with sets of structures $E, E'$. An everywhere biregular birational transformation $f$ of $V$ onto $V'$ is said to be an isomorphism of $(V, E)$ and $(V', E')$ if $f(E) = E'$. When $V = V', E = E'$, we say that $f$ is an automorphism.

It is well known that when $V$ is a curve, the set of automorphisms of it forms an algebraic group and, in particular, when the genus is greater than 1, it is actually a finite group. It is also well known that when $V$ is an Abelian variety of dimension 1, the set of automorphisms of it forms a finite group. But when $\dim V > 1$, the situation is different. In fact, there is a non-singular algebraic surface in a projective space such that its group of automorphisms is an infinite group but is not an algebraic group. These facts depend on the fact that a curve carries its uniquely

† We follow the same terminology and conventions as in Weil\cite{6,7}.
determined polarization, but when \( \dim V \geq 2 \), the same is no longer true in general. In fact we have the following theorem (cf. Matsusaka\(^{[2]}\)):

Let \((V, \mathcal{A})\) be a polarized variety, and let \( \mathcal{G} \) be the set of automorphisms of it, then \( \mathcal{G} \) is an algebraic group and the connected component \( \mathcal{G}_0 \) of the neutral element of \( \mathcal{G} \) is an extension of a subgroup of the Picard variety of \( V \) by a linear group.

Also we can show that when \( \mathcal{G}' \) is the set of all automorphisms of \( V \), then \( \mathcal{G}_0 \) is the largest connected algebraic group in \( \mathcal{G}' \). As an immediate corollary of the above theorem, we see that the set of automorphisms of a polarized Abelian variety is a finite group (cf. Weil\(^{[9]}\); Matsusaka\(^{[2]}\)).

It is seen that \( \mathcal{G}_0 \) is a normal subgroup of \( \mathcal{G}' \) and, as far as those examples which the author knows are concerned, \( \mathcal{G}'/\mathcal{G}_0 \) is a finitely generated group.

2. Equivalent projective embeddings; Moduli

Let \((V, \mathcal{A})\) be a polarized variety and let \( X, X' \) be two linearly effective divisors in \( \mathcal{A} \). Then \( X \) and \( X' \) determine uniquely projective embeddings \( f, f' \) of \( V \) up to projective transformations (here we consider only those projective embeddings which come from the linearly independent base of \( \mathcal{L}(X), \mathcal{L}(X') \)). We say that \( f \) and \( f' \) are equivalent embeddings in the broad sense. Moreover, when \( X \equiv X' \bmod G_a \), we say that they are equivalent in the strict sense.

Let \( \mathcal{G} \) be a subset of \( \mathcal{A} \); then by \( \mathcal{P}(V, \mathcal{G}) \) we understand the set of projective varieties \( f(V) \), where \( f \) is a projective embedding determined by a linearly effective divisor contained in \( \mathcal{G} \). \( \mathcal{X} \) can be written as \( \bigcup \mathcal{A}(X) \) where \( \mathcal{A}(X) \) is the coset of \( G_a \) containing \( X \), and except for a finite set, say \( \mathcal{A}(X_1), \ldots, \mathcal{A}(X_s) \), every component \( \mathcal{A}(X) \) is an irreducible algebraic family consisting of linearly effective divisors such that all divisors in it determine complete linear systems of the same dimension. From this we can deduce that \( \mathcal{P}(V, \mathcal{A}(X)) \), for such \( \mathcal{A}(X) \), has a structure of an open algebraic variety in a projective space, having the smallest field of definition (this can be done conveniently in terms of Chow-forms).

Put \( \mathcal{X}^* = \mathcal{X} - (X_1, \ldots, X_s) \). Then \( \mathcal{P}(V, \mathcal{X}^*) = \bigcup_{X \in \mathcal{X}^*} \mathcal{P}(V, \mathcal{A}(X)) \) and there is a smallest field \( K \) such that

(i) one of the \( \mathcal{P}(V, \mathcal{A}(X)) \) in the expression for \( \mathcal{P}(V, \mathcal{X}^*) \) is defined over \( K \),

(ii) every other \( \mathcal{P}(V, \mathcal{A}(X)) \) in the expression for \( \mathcal{P}(V, \mathcal{X}^*) \) is defined over a separably generated extension of \( K \), provided that the order of \( G_n/G_a \) is prime to the characteristic. We define this field \( K \) as the field of moduli.
of \((V, \mathcal{X})\). In the case of characteristic 0, \(K\) is the smallest field such that 
\((V^\varphi, \mathcal{X}^\varphi)\) is isomorphic to \((V, \mathcal{X})\), where \(\varphi\) is any automorphism of the 
universal domain over \(K\), as Shimura remarked.

In the case when \(V\) is an Abelian variety of dimension 1, the field of 
moduli \(K\) is generated over the prime field by the corresponding values 
of modular functions. The corresponding fact seems to be true in general 
when \(V\) is an Abelian variety and \(\mathcal{X}\) contains a divisor determined by 
the unimodular principal matrix of the Riemann matrix defining \(V\). 
For discussions of (ii), cf. Matsusaka\(^{[3]}\).

3. Torelli's theorem

It is known that even if two curves \(\Gamma\) and \(\Gamma'\) have isomorphic Jacobian 
varieties, they may not be birationally equivalent to each other. Let 
\(J\) and \(J'\) be Jacobian varieties of \(\Gamma\) and \(\Gamma'\), and let \(\theta\) and \(\theta'\) be canonical 
divisors on \(J\) and \(J'\) with respect to \(\Gamma\) and \(\Gamma'\). \(\theta\) and \(\theta'\) determine on \(J\) 
and \(J'\) polarizations (cf. Weil\(^{[8]}\)), which we call canonical polarizations 
with respect to \(\Gamma\) and \(\Gamma'\). Torelli's theorem asserts that \(\Gamma\) and \(\Gamma'\) are 
birationally equivalent to each other if and only if the canonically polarized 
\(J\) and \(J'\) are isomorphic. This theorem was first proved by Torelli (cf. 
Torelli\(^{[5]}\)), and recently proofs were given by Weil, Andreotti and myself 
which are valid over fields of any characteristic (cf. Weil\(^{[10]}\); Andreotti\(^{[1]}\); 
Matsusaka\(^{[3]}\)).

4. A characterization of canonically polarized Jacobian varieties

Let \((A, \mathcal{X})\) be a polarized Abelian variety. When \(A\) is of dimension 2, 
\((A, \mathcal{X})\) is a canonically polarized Jacobian variety if \(\mathcal{X}\) contains an irreducible divisor \(X\) such that \(\deg(X \cdot X_u) = 2\) (cf. Weil\(^{[18]}\)). The same fact 
seems to be true for 3-dimensional Abelian varieties. On the other hand, 
the above theorem is a special case of the following numerical criterion:

Let \((A, \mathcal{X})\) be a polarized Abelian variety of dimension \(n\); then it is a 
canonically polarized Jacobian variety if and only if \(\mathcal{X}\) contains an irreducible divisor \(X\) such that

\[
\begin{align*}
(i) & \quad (X_{u_1} \cdots X_{u_{n-1}}) \\
(ii) & \quad \deg(X_{u_1} \cdots X_{u_n}) = n! \\
\end{align*}
\]

is numerically equivalent to \((n - 1)! C\), where \(C\) is a positive 1-cycle on \(A\), 
and that

moreover, when this is so, \(C\) is irreducible, \(A\) is the Jacobian variety of \(C\), 
\(C\) is canonically embedded in \(A\), and \(X\) is a canonical divisor on \(A\) with 
respect to \(C\) (cf. Matsusaka\(^{[4]}\)).
This theorem was extended to the case when $X$ is reducible by W. Hoyt in his Chicago thesis, and he also studied a structure of a polarized Abelian variety which is a specialization of a canonically polarized Jacobian variety.

REFERENCES