I would like to discuss two results about finite groups:

(0) All finite groups of odd order are solvable.

(F) A finite group is nilpotent if it admits a fixed point free automorphism of prime order.

Walter Feit and I proved (0) after a prolonged joint effort [5]. A critical special case of (0) was proved by Walter Feit, Marshall Hall, Jr. and me [4]. A slightly stronger result than (F) is in the literature [16].

It was Suzuki who took the first big step toward the proof of (0), for he showed that the group \( G \) is solvable if it is of odd order and if in addition the centralizer of each non-identity element of \( G \) is abelian [15]. The proof of this apparently very special case removed one of the major stumbling-blocks in the proof of (0). Suzuki’s proof contains an application of exceptional characters. Exceptional characters were discovered by Brauer and Suzuki and their simplicity and power have been of great help in recent work in finite groups. Suzuki’s proof also presents a structure for the proof of (0). This structure is very easy to describe; it has two parts:

(1) Determination of the maximal subgroups.

(2) Exploitation of Frobenius reciprocity.

The special case of (0) alluded to consists simply in replacing “abelian” by “nilpotent” in Suzuki’s result. A skirmish with \( p \)-groups and a modification of one character argument mark the only divergence from Suzuki’s proof.

The proof of (0) involves a large number of technical problems and a great deal of case analysis. Proving (0) by induction on the order of \( G \), it can be assumed that \( G \) is simple and that every proper subgroup of \( G \) is solvable. In the determination of the maximal subgroups of \( G \), a much more serious contact with solvable groups is required than is customary. An initial analysis, to which I will return, shows that the maximal subgroups of \( G \) are of two types. Type 1 consists of those subgroups \( M \) which are split extensions of a nilpotent normal subgroup \( H \) by a group \( E \) with the property that \( E \) contains a subgroup \( E_0 \) of the same exponent as \( E \) such that non-identity elements of \( E_0 \) induce fixed point free automorphisms of \( H \). Furthermore, the set of elements of \( E \) which have non-identity fixed points on \( H \) lies in a normal abelian subgroup of \( E \). Type 2 consists of those subgroups \( M \) which are extensions of a normal Hall subgroup \( H \) by a non-identity subgroup \( E \) such that any two non-identity elements of \( E \) have the same cyclic fixed point set on \( H \). Furthermore, \( H \) is the derived group of \( M \). These conditions imply that the derived group of \( H \) is nilpotent, and if \( H \) is not nilpotent, then \( E \) has prime order.

If \( M \) is of type 1, \( H_0 \) denotes the set of elements of \( M \) commuting with some non-identity element of \( H \). If \( M \) is of type 2 and \( H \) is nilpotent, \( H = H_0 \), while if \( H \) is not nilpotent, \( H_0 \) is a suitable subset of \( H \). The set \( H_0 \) has many properties. Two elements of \( H_0 \) are \( G \)-conjugate only if they are \( M \)-conjugate. If \( x \) is a non-identity element of \( H_0 \) and if the centralizer of \( x \) is not contained in \( M \), then there is a maximal subgroup \( N \) of \( G \) containing the centralizer
of \( x \) with the property that \( N \cap M \) has a normal complement in \( N \). Also, the index of \( N \cap M \) in \( N \) is relatively prime to the order of every element of \( H_0 \) and to certain other relevant integers.

The proofs of the preceding statements involve the "serious contact with solvable groups" mentioned above. Assuming these results, let \( C \) be the set of class functions of \( M \) vanishing outside \( H_0 \) and let \( C_0 \) be the subset of \( C \) whose elements vanish also at the identity. There is a linear isometry \( t \) mapping \( C_0 \) into the class functions of \( G \), and \( t \) maps generalized characters into generalized characters. The existence of \( t \) is established by exhibiting explicitly suitable class functions of subgroups of \( G \) and checking that the preceding results render the induction map manageable. The mapping \( t \) is the bridge between local and global information, since if \( c \) is in \( C_0 \), then \( c \) and its image under \( t \) agree on \( H_0 \).

It is then necessary to attempt to extend \( t \) to \( C \). The basic idea here is Feit's [3] and with a suitable modification it is shown that \( t \) may be extended provided only that certain inequalities are satisfied. These inequalities depend only on the group \( M \) and not on its imbedding in \( G \). If these inequalities are not satisfied, the structure of \( M \) is very limited.

The most delicate portion of the above analysis deals with a subgroup \( M \) of type 2, where \( H \) is a non-abelian \( p \)-group. It is a critical part of the proof to show that even if the relevant inequalities are violated, the desired extension of \( t \) is still available. Success is obtained only by considering a suitable maximal subgroup of \( G \) which contains \( E \) and is not conjugate to \( M \), that is, it is necessary to study the imbedding of \( M \) in \( G \).

Once the problem of extending \( t \) has been exhaustively examined, several striking results emerge. If \( M \) is of type 1, then \( E = E_0 \), that is, \( M \) is a Frobenius group. If \( M \) is of type 2, then \( H \) is not nilpotent, so that \( E \) is of prime order. In conjunction with the special case of (0) already mentioned, it follows that \( G \) contains two non conjugate maximal subgroups \( S \) and \( T \), each of type 2, whose intersection is a self-normalizing cyclic subgroup of order \( pq \), \( p \) and \( q \) being distinct primes. Furthermore, both \( S' \) and \( S/S'' \) are Frobenius groups.

In this bizarre situation, relentless pursuit of inequalities involving sets of exceptional characters shows that both \( T' \) and \( T/T'' \) are Frobenius groups, and the structure of \( S \) and \( T \) is determined explicitly. \( S \) is isomorphic to the group of mappings \( m_{a,b,s}: x \mapsto ax^s+b \), where \( a, x, b \) are in \( F(p^s) \), the field of \( p^s \) elements, \( N(a) = a^{1+p^s+\ldots+p^{s-1}} = 1 \), and where \( s \) ranges over the automorphisms of \( F(p^s) \). \( T \) is obtained from \( S \) by interchanging \( p \) and \( q \).

The symmetry between \( S \) and \( T \) is now destroyed by taking \( p > q \). In this situation, \( T \) contains a cyclic subgroup \( V \) of order \( (q^p-1)/(q-1) \) which is the centralizer of each of its non-identity elements, and \( V \) is of index \( pq \) in its normalizer. On the other hand, \( S \) contains a cyclic subgroup \( U \) of order \( (p^q-1)/(p-1) \). Furthermore, either \( U \) is the centralizer of each of its non-identity elements and \( U \) is of index \( pq \) in its normalizer, or \( U \) is conjugate to a subgroup of \( V \). The first possibility can be handled in a fairly straightforward fashion, since the exceptional characters for \( U \) are distinct from those for \( V \). In the second case, however, the exceptional characters for \( U \) and \( V \) coalesce, and the character theory ceases to produce further reductions. Of course, the information that \( (p^q-1)/(p-1) \) divides \( (q^p-1)/(q-1) \) is now available since \( U \) is conjugate to a subgroup of \( V \), but this information is difficult to exploit.
The final analysis involves the study of generators and relations which capture the intertwined nature of $S$ and $T$ and leads to a question about $F(p^s)$. Namely, if $f$ is the linear fractional transformation sending $x$ to $\frac{2-x}{1}$, does there exist an element $a$ of $F(p^s)$ different from 1 such that $N(a^s) = 1$ for all powers $s$ of $f$? This question is considerably easier than the previous divisibility problem, and the negative answer retranslates to the relations to give a contradiction and completes the proof of (0).

The problems involving solvable groups which arise in the proof of (0) are diverse. The arguments are based on P. Hall’s theory of solvable groups [6–13]. The very exploitable Theorem B of Hall and Higman [14] also plays an important role. N. Blackburn has classified all $p$-groups which contain no normal elementary abelian subgroup of order $p^3$, $p$ being an odd prime [1, 2]. Such $p$-groups contain no elementary abelian subgroup of order $p^3$. It is an easy consequence of this result that if $P$ is such a $p$-group and $P$ is a $S_p$-subgroup of the $p$-solvable group $H$, and $H$ is of odd order, then $H$ has $p$-length one, and elements of $P$ are $H$-conjugate only if they are conjugate in the normalizer of $P$. Thus, for many purposes, we can consider the imbedding in $G$ of such $S_p$-subgroups as known.

Let $s$ denote the set of primes $p$ such that $G$ contains an elementary abelian subgroup of order $p^3$. If $p$ is in $s$ and $P$ is a $S_p$-subgroup of $G$ let $A$ be a maximal normal abelian subgroup of $P$ which is not generated by two elements. The first major reduction in the proof of (0) is that for each prime $q$ different from $p$, the centralizer of $A$ permutes transitively by conjugation those $q$-subgroups of $G$ which are maximal with respect to the property of being normalized by $A$. Under the hypothesis that $A$ normalizes some non-identity $q$-subgroup of $G$, the preceding result has a powerful consequence. Namely, if $N$ is the normalizer of the center of the weak closure of $A$ in $P$, then $N$ has no normal subgroup of index $p$, and if $H$ is any subgroup of $P$ which contains $A$, then the normalizer of $H$ is contained in $N$.

If $p_1$ and $p_2$ are in $s$, write $p_1*p_2$ provided $G$ contains subgroups $E_1, E_2$ which are elementary abelian of orders $p_1^3, p_2^3$ respectively, such that $E_1$ and $E_2$ generate a proper subgroup of $G$. The next reduction states that if $p_1*p_2$ and if $P_1$ is a $S_{p_1}$-subgroup of $G$, and if $P_1$ centralizes every $p_2$-subgroup of $G$ which it normalizes and $P_2$ centralizes every $p_1$-subgroup of $G$ which it normalizes, then $G$ contains a nilpotent $S_{p_1*p_2}$-subgroup.

A variety of further reductions leads to a proof that $*$ is an equivalence relation on $s$ and that if $e$ is an equivalence class, then $G$ contains a $S_e$-subgroup $H$, while $H \cap H^e$ is of square free order for every $x$ in $G$ which does not normalize $H$. The method of proof shows that $H$ is a proper subgroup of $G$. The transition from this result to the more precise structure and imbedding in $G$ of the maximal subgroups of $G$ is somewhat technical but straightforward.

It is now possible to indicate a major difficulty in the study of the subgroups of $G$ and at the same time to indicate the connection between this difficulty and the proof of (F). In proving (F), it was convenient to prove a stronger result. The nature of the induction hypothesis for this stronger result permits the problem to be reduced to a detailed examination of a suitable solvable subgroup $K$ of the group in question. The group $K$ is of order $p^aq^b$ and has the property that if $H$ is the maximal normal $p$-subgroup of $K$ and $Q$ is the maximal normal $q$-subgroup of $K$, then $H$ maps onto the maximal normal $p$-subgroup of $K/Q$. Furthermore, $K$ contains a $S_p$-sub-
group of the normalizer of $H$. This is exactly the situation encountered in the proof of (0). In such a situation, it can be shown that if $P$ is an $S_p$-subgroup of $K$ contained in the $S_p$-subgroup $P^*$ of $G$, then $H$ contains every normal abelian subgroup of $P^*$. Let $A$ be any maximal normal abelian subgroup of $P^*$ and let $C$ be any conjugate of $A$ contained in $K$. It is important to conclude that $C$ is contained in $H$, that is, Wielandt's notion of weak closure plays a dominant role here [17]. To obtain this containment, Theorem B of Hall-Higman [14] is applied. If $A$ is not elementary, Theorem B suffices for (F). However, if $A$ is elementary, the argument breaks down in (F), while in (0) it is only possible to assert that $K$ is generated by $K \cap N_1$ and $K \cap N_2$, where $N_1$ and $N_2$ are subgroups of $G$ whose definition is independent of $K$. Even this weaker result is quite powerful and in the proof of (0) the systematic exploitation of this fact leads eventually to the containment $C \subseteq H$. The argument succeeds because there are many subgroups which can play the role of $A$. It is not so important that $A$ be a maximal normal abelian subgroup of $P^*$. The relevant point is that $A$ be a normal subgroup of $P^*$ which contains all elements of $P^*$ which centralize $A$, and the class of nilpotency of $A$ must be small enough to guarantee that $A \subseteq H$. In fact, even these conditions may be relaxed somewhat. This latitude in the choice of $A$ plays a critical role in the proof of both (0) and (F).

In the proof of (0), the hypothesis that $G$ is of odd order is used in a variety of ways. Theorem B is sharper for groups of odd order than for groups of even order. Various results about $p$-groups fail to hold for $p=2$. Non-principal irreducible characters of groups of odd order are non-real. The analysis of characters leads to inequalities in which the parameter is a prime divisor of the group order, and the inequalities force the prime to be small. If $S$ is a group of odd order, and if $S$ does not contain an elementary abelian subgroup of order $p^3$ for any prime $p$, then $S'$ is nilpotent. This easy result is used frequently. The symmetric group on four letters is a counterexample if the oddness condition is omitted.

It is too early to indicate whether the techniques used in proving (0) and (F) will be relevant in the further study of finite groups. It is certain that these techniques can be refined to give a neater proof of (0) than the present one. Perhaps in conjunction with Brauer's modular theory, new results can be obtained. Also, it is reasonable that the cohomology groups of finite groups will yield insights denied to the more "internal" analysis characterizing the proofs of (0) and (F). Finally, we can hope that involutions enjoy properties which are yet unexploited.

REFERENCES


[5]. Feit, W. & Thompson, J. G., Solvability of Groups of Odd Order. (Mimeographed notes.)


[13]. ——— Lecture notes.


