§1. The method of Oka

The history of the Levi problem is described in the lecture of H. Grauert in this Congress and I shall not repeat it. We shall rather analyze the work of Oka [6] on this problem, and see how the underlying ideas are susceptible of generalization.

Let $D$ be an unramified domain over $C^n$, i.e. a connected complex manifold of dimension $n$ together with a holomorphic map $\varphi: D \to C^n$ whose jacobian is nowhere zero. We say that $D$ "satisfies the Hartogs continuity condition" if the following holds.

Every point of the relative boundary of $D$ over $C^n$ (which depends on $\varphi$) has a neighbourhood $U$ such that if $\sigma_t, 0 < t < 1$ is a continuous family of complex 1-dimensional discs in $U$ and $U(\sigma_t) \subseteq U$, then $\cup \sigma_t \subseteq U$.

Here $A \subseteq X$ means that $A$ is relatively compact in $X$.

A complex space $X$ is holomorph-convex if for any infinite discrete set $\{x_v\}$, there is a holomorphic function $f$ on $x$ for which $f(x_v)$ is unbounded. If in addition holomorphic functions separate points, $X$ is called a Stein space, (and we say that $X$ is Stein).

The main theorem of Oka is then as follows.

**Theorem 1.** An unramified domain over $C^n$ which satisfies the Hartogs continuity condition is a Stein space.

The proof of Oka splits into several parts. To describe this proof, and for later use, we require the concept of pseudoconvex functions [6] (more often called "plurisubharmonic functions" in the literature).

**Definition.** An upper semicontinuous function $p$ on an open set $\Omega \subseteq C^n$ is pseudoconvex if its restriction to any complex disc in $\Omega$ is subharmonic. It is strongly pseudoconvex (or just strongly convex) if to every open $\Omega' \subseteq \Omega$ and any $C^\infty$ function $h$ on $\Omega$, there is $\varepsilon > 0$ such that $p + \varepsilon h$ is pseudoconvex on $\Omega'$.

We remark for later use that the definition can be extended to complex spaces by means of a local imbedding of the space into $C^n$.

Oka first proves the following.

**Lemma 1.** If $D$ satisfies the continuity condition and for $x \in D$, $d(x) = d_\varphi(x)$ denotes the Euclidean radius of the largest univalent ball about $x$ in $D$, then $-\log d(x)$ is pseudoconvex in $D$.

Using this, he proves

**Lemma 2.** If $D$ is as above, then there exists a strongly convex function $p$ such that for any $\alpha > 0$, the set...
is relatively compact in $D$. Moreover, $p$ can be chosen so that locally it can be represented as the maximum of finitely many $C^\infty$ strongly convex functions.

Using the existence of this function $p$ on any domain which satisfies the continuity theorem, he proves

**Theorem 2.** Let $(D, \varphi)$, $\varphi = (\varphi_1, \ldots, \varphi_n)$ be an unramified domain over $C^n$. Let $\delta > 0$ and $D_1(\delta) = \{x \in D | \text{Re } \varphi_1 < +\delta\}$, $D_2(\delta) = \{x \in D | \text{Re } \varphi_1 > -\delta\}$.
Suppose that for some $\delta > 0$, $D_1(\delta)$ and $D_2(\delta)$ are Stein. Then $D$ is Stein.

It is a simple matter to deduce Theorem 1 from Lemma 2 and Theorem 2. In the proof of this theorem, Oka introduced a principle which, after Grauert [3], can be formulated as follows.

A complex space such that for any $x_0 \in X$, there is a holomorphic map $f : X \to C^k$ such that $x_0$ is an isolated point of $f^{-1} f(x_0)$ is called $K$-complete.

**Theorem 3.** Let $X$ be a holomorph-convex, $K$-complete space. Then global holomorphic functions separate points and give local coordinates at every point of $X$.

I would like to say that every published proof of the theorems concerning the Levi problem described in this article depend, more or less explicitly, on this principle.

§2. The Levi problem for abstract spaces

The methods of Oka make heavy use of the fact that one deals with domains over $C^n$. However, following an idea of Grauert [4], it is possible to generalize Oka's theorem to arbitrary complex spaces. (See [5]).

**Theorem 4.** Let $X$ be a complex space and $p$ a continuous strongly convex function such that for any $\alpha > 0$, the set

$$\{x \in X | p(x) < \alpha\} \ll X.$$

Then $X$ is a Stein space. On a Stein space, the function $p$ can be chosen to be, in addition, real analytic.

For domains $D$ on a complex space $X$ with $D \ll X$, it is possible to obtain a theorem where the condition on the boundary is more local [5].

**Definition.** Let $D \ll X$ be an open set on the complex space $X$. $D$ is called pseudoconvex (strongly pseudoconvex) if to every $x_0 \in \partial D$ there is a neighbourhood $U$ and in $U$ a continuous pseudoconvex function (strongly convex function) $p$ with

$$U \cap D = \{x \in U | p(x) < 0\}.$$

**Theorem 5.** Any strongly pseudoconvex domain on a complex space is holomorph-convex and obtained from a Stein space by "blowing up" finitely many points.

Results of the type of Theorem 4, and Oka's theorem, are established by proving first the holomorph-convexity of certain relatively compact sub-
domains defined as in Theorem 5 and then applying an approximation argument to pass to the whole space. Perhaps the most typical of these approximation theorems is the following, (see [5]).

**Theorem 6.** Let $X$ be a Stein space and $p$ a continuous pseudoconvex function on $X$. Then, for any real $\alpha$, the set

$$\{x \in X | p(x) < \alpha\} = X_\alpha$$

is a Stein space which is Runge in $X$, i.e. holomorphic functions on $X_\alpha$ can be approximated by holomorphic functions on $X$.

This theorem, when $X$ is an unramified domain over $\mathbb{C}^n$ was proved by H. Behnke and K. Stein [1]. When $X$ is a Stein manifold F. Docquier and H. Grauert [2] reduced the problem to the case of domains in $\mathbb{C}^n$. These methods require one to prove first that $X_\alpha$ is a Stein space. The general case requires, however, a quite different method, which proves that $X_\alpha$ is Stein at the same time as proving that it is Runge in $X$ [5]. I would like to add that the idea of this method is contained in the work of Oka.

The condition of *strong* pseudoconvexity in Theorems 4 and 5 is essential as is shown by an example of Grauert (see below). Very little is known about pseudoconvex domains which are not strongly pseudoconvex. It is perhaps possible that at least for domains with smooth boundary on a complex manifold, pseudoconvexity of the domain, together with *strong* pseudoconvexity at one point, imply its holomorph-convexity. We are however far from a proof of such a theorem.

Docquier and Grauert [2] showed that for unramified domains over Stein manifolds $X$, the analogue of Oka's Theorem 1 is true and their proof is in fact a reduction of the problem to Theorem 1 by imbedding $X$ into $\mathbb{C}^n$ and showing that there is a neighbourhood $U$ of $X$ in $\mathbb{C}^n$ and a holomorphic map $U \rightarrow X$ which is the identity on $X$. This is not true even if $X$ has one singular point, and it is not known even if pseudoconvex domains on Stein spaces are themselves Stein. The best theorem that has been proved in this direction is the following theorem, obtained in collaboration with A. Andreotti.

**Theorem 7.** If $X$ is $K$-complete and $p$ a continuous pseudoconvex function with $\{x \in X | p(x) < \alpha\} \subseteq X$ for any $\alpha > 0$, then $X$ is Stein.

Another outstanding problem is the analogue of Theorem 1 for ramified domains over $\mathbb{C}^n$.

The example given by Grauert of a pseudoconvex domain that is not holomorph-convex is as follows.

Let $n > 1$ and let $\Gamma$ be the lattice in $\mathbb{C}^n$ generated by $w_1 = (1,0,...,0),$ $w_j = (w_{j1},...,w_{jn})$, $j = 2,...,2n$ with $w_j = w_j' + i w_j''$. Suppose that $w_1,...,w_{2n}$ are $R$-independent, $w_j = 0$ for $j \geq 2$, and $w_{j1}$ ($j = 2,...,2n$) are linearly independent over the integers.

Let $T^n$ be the torus $\mathbb{C}^n/\Gamma\cdot 2: \mathbb{C}^n \rightarrow T^n$ the natural map. Let $U \subseteq \mathbb{C}^n$ be defined by $0 < \text{Re} z_1 < \frac{1}{2}$. Let

$$\varphi(z) = \frac{1}{1 - 2 \text{Re} z_1} + \frac{1}{\text{Re} z_1}.$$
Then \( \varphi \) is invariant under any \( \gamma \in \Gamma \) with \( (\gamma \hat{U}) \cap \hat{U} \neq \emptyset \) and so defines a function \( p \) on \( D=\pi(U) \). Clearly \( p \to \infty \) at the boundary of \( D \) and is pseudo-convex. But every holomorphic function \( f \) on \( D \) is constant: in fact \( |f| \) has a maximum at a point \( x_0 \) on \( K=\pi(\text{Re } z_1 = \frac{1}{2}) \) and there is a connected \( n-1 \) dimensional analytic set \( \mathcal{A} \) through \( x_0 \) in \( K \) which is dense in \( K \) (viz. \( \pi(z_1 = c_1) \)), where \( c \) is such that \( \pi(c) = x_0 \). Hence \( f \) is constant on \( \mathcal{A} \) and so on \( K \). Since \( K \) has real dimension \( 2n-1 \), \( f \) is constant on \( D \).

REFERENCES


