THE GROTHENDIECK RING IN GEOMETRY AND TOPOLOGY

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§ 1. The Grothendieck ring in homotopy theory

I am going to be talking about vector bundles, i.e. fibre bundles with fibre a vector space and group the linear group. Vector bundles are to the geometer what representations or modules are to the algebraist. In fact the modern algebraic geometer hardly distinguishes between the two. Now, in the study of groups, representation theory is important for two reasons: (*a*) because representations of groups occur naturally in many contexts, and (*b*) because characters provide a powerful technical tool with which to attack groups.

The same essentially holds for vector bundles. That they occur naturally in many geometrical problems is of course well known. It is less obvious, but equally true, that, when formalized into the Grothendieck ring, they provide a powerful tool in homotopy theory. The Grothendieck ring is a very simple object which can be defined in a wide range of categories. In the classical representation theory of finite groups one obtains the ring of generalized characters (whose elements are integral linear combinations of the irreducible characters). For vector bundles the definition is as follows.

Let X be a "reasonable" space (say a polyhedron) and consider complex(1) vector bundles over X. If E, F are two vector bundles then we can form their direct sum $E \oplus F$ and their tensor product $E \otimes F$. The operation $\oplus$ turns the set $\mathcal{E}(X)$ of isomorphism classes of vector bundles into an abelian semi-group. $K(X)$ is defined to be the associated abelian group. This means that $K(X)$ is an abelian group with a homomorphism $\theta: \mathcal{E}(X) \to K(X)$ such that

(i) Im (θ) generates $K(X)$,
(ii) $\theta(E) = \theta(F)$ if and only if $\phi(E) = \phi(F)$ for every homomorphism ϕ of $\mathcal{E}(X)$ into an abelian group.

It is elementary to show that $K(X)$, θ exist and that (i), (ii) make them essentially unique. The operation $\otimes$ then turns $K(X)$ into a commutative ring—the Grothendieck ring. It is a contravariant functor of X.

Now character theory is essentially a comparison of a given group with the linear groups, and depends for its success on our detailed knowledge of the linear groups. In the same way the study of $K(X)$ is essentially a comparison, in the sense of homotopy, of X with the linear groups (or rather their classifying spaces, the Grassmannians), and depends for its success on detailed knowledge of the homotopy of the linear groups. This essential information is provided by Bott's periodicity theorem which, in our formulation, asserts [5];

$$K(X \times S^2) \cong K(X) \otimes_Z K(S^2), \tag{1}$$

(1) As for representations the complex case is simpler but the real theory can also be developed.

where S^2 is the 2-sphere. This basic formula really states that the (stable) homotopy of the complex linear group is just what linear algebra dictates. For this reason the general machinery which one can develop from (1) is very simple and enables K to be handled with great ease. For instance the computation of $K(X)$ for many homogeneous spaces turns out to be a fairly easy matter (cf. [2]).

The formal properties of K are similar in many ways to those of H (cohomology) and it can be applied in a similar way in problems of homotopy theory. Cohomology operations ($H \to H$) can be replaced either by characteristic classes ($K \to H$) or by operations ($K \to K$). It was pointed out by Grothendieck that one can very easily define operations ($K \to K$) by using operations on vector spaces, and this has been exploited by Adams(1) [1] in his solution of the long-outstanding problem of vector-fields on spheres.

I do not propose to discuss these questions in any further detail, but I will just make a few general remarks about the method. In orthodox algebraic topology one breaks a space down into atomic parts (cohomology or homotopy) and then tries to see how the atoms were put together (operations or k-invariants). Now many of the spaces that turn up in practice, e.g. homogeneous spaces, have such a regular structure that it is a pity to break them right down. Instead, like the biochemist, we should look around for large standard molecules out of which these spaces are built. It would seem that the linear groups provide such standard molecules and that this, in a philosophical sense, explains the success of K in homotopy theory.

§ 2. Geometric applications

Because vector bundles arise so naturally it is to be expected that K, besides its general use in homotopy theory, will have special relevance in more geometric contexts. In order to describe some examples I must now explain one technical point.

If Y is a subspace of X then one can define a relative group $K(X, Y)$ by considering vector bundles on X with a given trivialization on Y. Suppose now that E, F are two vector bundles on X and that $\alpha: E \to F$ is an isomorphism on Y. Then by adding to both E and F a "complement" of F (i.e. an F' so that $F \oplus F'$ is trivial) we get an element

$$d(E, F, \alpha) \in K(X, Y)$$

called the "difference bundle", and this is independent of the choice of F'. I shall now discuss three examples in which this "difference bundle" arises naturally and leads to interesting results.

§ 2.1. Sub-manifolds

Let X be a compact oriented n-manifold embedded in S^{n+2k}. Choose a metric, let N be a tubular neighbourhood of X and put $N_0 = N - X$. The normal bundle ν has structure group $SO(2k)$ but when lifted up to N_0 this

(1) Actually Adams uses K for real vector bundles. Also his operations are a modification of those introduced by Grothendieck.

reduces to $SO(2k-1)$. Now let V be any representation space of $SO(2k)$, and let σ be the outer automorphism of $SO(2k)$ corresponding to the reflection in the plane $x_{2k}=0$. Then V^σ will be a new representation of $SO(2k)$ but, since σ is the identity on $SO(2k-1)$, V and V^σ coincide as representations of $SO(2k-1)$. Take the vector bundles E, E^σ associated to ν (lifted up to N) by the representations V, V^σ. Then on N_0 we have a canonical isomorphism $\alpha: E \to E^\sigma$ and hence a difference bundle

$$d(E, E^\sigma, \alpha) \in K(N, N_0).$$

Using the natural map

$$K(N, N_0) \to K(S^{n+2k})$$

we end up with an element of $K(S^{n+2k})$. But we know all about this group; in particular we know all about characteristic classes on S^{n+2k}. From this we deduce information about the characteristic numbers of X. These results are of two types:

(i) (stable) since X can always be embedded in S^{n+2k} for large k we get some results for any X; these are of importance for example in the study of differentiable structures on manifolds;
(ii) (unstable) taking a low value of k may, for given X, give results which are false; this shows that X cannot be embedded in S^{n+2k} (or $\mathbf{R}^{n+2k}$).

An example of (ii) is the following result [3]:
Complex projective n-space cannot be embedded differentiably in $\mathbf{R}^{4n-2\alpha(n)}$, *where* $\alpha(n)$ *is the number of* 1's *in the dyadic expansion of* n.

§ 2.2. Sheaves

Let X be a complex manifold, S a coherent analytic sheaf on X and suppose we have an exact sequence of coherent sheaves:

$$0 \to E \xrightarrow{\alpha} F \to S \to 0, \tag{2}$$

where E, F are locally free. Then E, F are germs of holomorphic sections of vector bundles (still denoted by E, F) and α gives a vector bundle homomorphism (still denoted by α). Outside $|S| = \text{support}(S)$ α is an isomorphism, and hence we get a difference bundle,

$$d(E, F, \alpha) \in K(X, X - |S|). \tag{3}$$

More generally if, instead of (2), we have a (finite) resolution $\mathfrak{L}$ of S by locally free sheaves then we can generalize (3) and define an element

$$d(\mathfrak{L}) \in K(X, X - |S|), \tag{4}$$

which turns out to depend only on S and not on $\mathfrak{L}$. We can apply this in particular when S is the sheaf of a complex analytic subvariety of X. Using the element given by (4) and the relations between K and H one can prove the following [4]:(1)

If $u \in H^{2k}(X; \mathbf{Z})$ *is represented by a complex analytic subvariety (of codimension* k*) then* $Sq^3 u = 0$.

Here Sq^3 is the Steenrod operation

$$H^{2k}(X; \mathbf{Z}) \rightarrow H^{2k+3}(X; \mathbf{Z}).$$

This result is of interest because it disproves an old conjecture of Hodge.

§ 2.3. Elliptic operators

Let X be a compact oriented C^∞-manifold, E, F two C^∞ n-dimensional complex vector bundles over X, $\Gamma(E)$ and $\Gamma(F)$ the spaces of C^∞ cross-sections. Then by a partial differential operator from E to F one means a linear operator

$$D : \Gamma(E) \rightarrow \Gamma(F),$$

which in local coordinates is expressed by a matrix of partial derivatives. If we consider only the highest(2) order derivatives and regard $\partial/\partial x_i$ not as an operator but as a coordinate in the bundle T of cotangent vectors of X, we get a well-defined homomorphism

$$\sigma(D) : \pi^*(E) \rightarrow \pi^*(F),$$

where $\pi : T \rightarrow X$ is the bundle projection. This is called the symbol of the operator D, and D is said to be *elliptic* if $\sigma(D)$ is an isomorphism on T_0, the set of non-zero tangent vectors. Hence an elliptic operator defines a difference bundle

$$\gamma(D) = d(\pi^* E, \pi^* F, \sigma(D)) \in K(T, T_0).$$

It appears that the interesting invariants of D are actually invariants of $\gamma(D)$. More specifically D has an invariant called the index defined by

$$\nu(D) = \dim \operatorname{Ker} D - \dim \operatorname{Coker} D$$

and it seems reasonable to conjecture that $\nu(D)$ is expressible by a universal polynomial in terms of the characteristic classes of $\gamma(D)$ and those of X. This is closely related to the Riemann–Roch theorem.

I should just end by saying that most of this is joint work with F. Hirzebruch, while the remarks in § 2.3 are based on some current work, not yet completed[3], with I. M. Singer.

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(1) This also follows, if X is algebraic, from the resolution of singularities (Hironaka). In fact one then has $Sq^r u = 0$ for all odd r.

(2) Precisely we take derivatives of order k if, somewhere on X, D involves derivatives of order k, but nowhere does D involve derivatives of order $k+1$.

(3) (Added in proof.) See a forthcoming note in the Bulletin of the American Mathematical Society.

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