MODEL THEORY

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1. Introduction

Twenty years ago, A. Robinson and A. Tarski lectured on the subject of model theory to the International Congress at Cambridge, Massachusetts. At that time the subject was just beginning, and only two real theorems were known. Since then progress has been so spectacular that today it takes years of graduate study to reach the frontier. In this lecture I will try to give an idea of what the subject is like and where it is going.

Model theory is a combination of universal algebra and logic. We start with a set $L$ of symbols for operations, constants, and relations, called a language; for example, $L = \{ +, \cdot, 0, 1, < \}$. The language $L$ is assumed to be finite or countable except when we specify otherwise. A model $\mathcal{M}$ for the language $L$ is an object of the form

$$\mathcal{M} = \langle A, +_{\mathcal{M}}, \cdot_{\mathcal{M}}, 0_{\mathcal{M}}, 1_{\mathcal{M}}, <_{\mathcal{M}} \rangle.$$ 

$A$ is a non-empty set, called the set of elements of $\mathcal{M}$, $+_{\mathcal{M}}$ and $\cdot_{\mathcal{M}}$ are binary operations on $A \times A$ into $A$, $0_{\mathcal{M}}$ and $1_{\mathcal{M}}$ are elements of $A$, and $<_{\mathcal{M}}$ is a binary relation on $A$.

EXAMPLES. — The field of rationals, $\langle \mathbb{Q}, +, \cdot, 0, 1 \rangle$, is a model for the language $\{ +, \cdot, 0, 1 \}$. So is every other ring, lattice with endpoints, etc. The ordered field $\langle \mathbb{Q}, +, \cdot, 0, 1, < \rangle$ is a model for the language $\{ +, \cdot, 0, 1, < \}$. Each group, partially ordered set, graph, etc., is a model for the appropriate language.

Most results in model theory apply to an arbitrary language. We frequently shift from one language to another, for instance a new theorem about a given language is often proved by applying an old theorem to a different language.

Many facts about models can be expressed in first order logic. In addition to the operation, relation, and constant symbols of $L$, first order logic has an infinite list of variables $x, y, z, v_0, v_1, v_2, \ldots$.

the equality symbol $=$, the connectives

$$\land \ (\text{and}), \ \lor \ (\text{or}), \ \neg \ (\text{not}),$$

and the quantifiers

$$\forall \ (\text{for all}), \ \exists \ (\text{there exists}).$$

Certain finite sequences of symbols are counted as terms, formulas, and sentences. The class of terms is defined as follows:

Every variable or constant is a term;

If $t, u$ are terms, so are $t + u, t \cdot u$. 
The formulas are defined by the rules:

If $t, u$ are terms, then $t = u, t < u$ are formulas.

If $\varphi, \psi$ are formulas and $v$ is a variable, then $\neg \varphi, \varphi \land \psi, \varphi \lor \psi, \forall v \varphi, \exists v \varphi$ are formulas.

A sentence is a formula all of whose variables are bound by quantifiers. For example, the sentence

$$(1) \forall x \ (x = 0 \lor \exists y \ (x \cdot y = 1))$$

states that every non-zero element has a right inverse.

Hereafter $\mathfrak{A} = \langle A, \ldots \rangle, \mathfrak{B} = \langle B, \ldots \rangle, \ldots$ denote models for $L$, and $\varphi, \psi, \theta, \ldots$ denote sentences.

The central notion in model theory is that of a sentence $\varphi$ being true in a model $\mathfrak{A}$, in symbols $\mathfrak{A} \models \varphi$. This relation between models and sentences is defined mathematically by an induction on the subformulas of $\varphi$. It coincides exactly with the intuitive concept. For example, the sentence (1) is true in the field of rationals but not in the ring of integers. A set of sentences is called a theory. $\mathfrak{A}$ is a model of a theory $T$, in symbols $\mathfrak{A} \models T$, if every sentence $\varphi \in T$ is true in $\mathfrak{A}$.

Examples. — The theory of rings is the familiar finite list of ring axioms found in any modern algebra text, and each ring is a model of this theory. The theory of real closed ordered fields is an infinite set of sentences, consisting of the axioms for ordered fields, the axiom stating that every positive element has a square root, and for each odd $n$ an axiom stating that every polynomial of degree $n$ has a root.

For each model $\mathfrak{A}$, the theory of $\mathfrak{A}$, $\text{Th}(\mathfrak{A})$, is the set of all sentences true in $\mathfrak{A}$.

Model theory is a rich subject which studies the interplay between various kinds of sentences and various kinds of models.

2. Two classical theorems.

Model theory traces its beginnings to two basic theorems which come out of the 1930's. The mathematicians who proved them are the founders of the subject.

Compactness Theorem. — If every finite subset of a set $T$ of sentences has a model, then $T$ has a model.

This theorem was first proved by Gödel, 1930 for countable languages. Malcev, 1936 extended the theorem to the case where $T$ is a set of sentences in an uncountable language. The compactness theorem has many applications to algebra (see Robinson, 1963).

Example. — Suppose the sentence $\varphi$ is true in every field of characteristic zero. Then there is an $n$ such that $\varphi$ is true in all fields of characteristic $p > n$.

Proof. — Consider the set $T$ of sentences consisting of the field axioms, the sentence $\neg \varphi$, and the infinite set

$$\neg (1 + 1 = 0), \neg (1 + 1 + 1 = 0), \neg (1 + 1 + 1 + 1 = 0), \ldots$$
By the hypothesis, \( T \) has no models, so some finite subset of \( T \) has no models, and the conclusion follows.

By the cardinal of a model \( \mathcal{A} \) we mean the cardinal of the set \( A \) of elements of \( \mathcal{A} \).

Löwenheim-Skolem-Tarski theorem. — If \( T \) has at least one infinite model, then \( T \) has a model of every infinite cardinality.

Example. — Let \( T \) be the theory of real closed fields. Then \( T \) has a model of cardinal \( 2^{\aleph_0} \), namely the field of real numbers. There are countable real closed fields and also real closed fields of every other infinite cardinality. The LST theorem shows that this happens in general.

Both of the theorems above assert that a certain kind of model exists, and their proofs depend on techniques for constructing models. Indeed, almost all the deeper results in model theory depend on the construction of a model. We shall indicate some of the most useful methods of constructing models and state some of the theorems which they yield.

3. The method of diagrams.

This method, due to Henkin, 1949 and Robinson, 1951, is the basis of Henkin's proof of the Gödel completeness theorem. It also has many other uses.

The diagram language for \( \mathcal{A} \) is obtained by adding to \( L \) a new constant symbol \( \bar{a} \) for each element \( a \) of \( A \). The elementary diagram of \( \mathcal{A} \), denoted by \( \text{Diag}(\mathcal{A}) \), is the set of all sentences in the diagram language of \( \mathcal{A} \) which are true in \( \mathcal{A} \). The difference between \( \text{Th}(\mathcal{A}) \) and \( \text{Diag}(\mathcal{A}) \) is that \( \text{Diag}(\mathcal{A}) \) has new constant symbols for the elements of \( \mathcal{A} \) while \( \text{Th}(\mathcal{A}) \) does not.

In many situations it is possible to construct a model of a set \( T \) of sentences by extending \( T \) to a set of sentences \( T' \) which happens to be an elementary diagram of some model \( \mathcal{A} \). In this construction one is always working with sentences, and constant symbols are used for the elements of \( \mathcal{A} \). The compactness and LST theorems can be proved by this method. The construction has many other applications; we shall state three of them without proofs.

The notation \( \varphi \models \psi \) means that every model of \( \varphi \) is a model of \( \psi \).

Theorem 1 (Craig interpolation theorem, Craig, 1957, A. Robinson, 1956). — Suppose \( \varphi \models \psi \). Then there is a sentence \( \theta \) such that \( \varphi \models \theta \), \( \theta \models \psi \), and every operation, constant, or relation symbol which occurs in \( \theta \) occurs in both \( \varphi \) and \( \psi \).

The next theorem concerns homomorphisms. A mapping \( h \) of \( A \) onto \( B \) is called a homomorphism, and \( \mathcal{B} \) is called the homomorphic image of \( \mathcal{A} \) by \( h \), if for all \( a, b, c \in A \),

\[
\begin{align*}
h(a +_A b) &= h(a) +_B h(b), \\
h(1_A) &= 1_B, \\
a <_A b \text{ implies } h(a) <_B h(b),
\end{align*}
\]

etc. If \( h \) is one-one and \( h^{-1} \) is also a homomorphism, then \( h \) is called an isomorphism. It is obvious that every sentence \( \varphi \) is preserved under isomorphic images, that is, every isomorphic image of a model of \( \varphi \) is a model of \( \varphi \). But which sentences are preserved under homomorphic images?
A sentence \( \varphi \) is said to be positive if it contains no negation symbol \( \neg \), i.e. it is built using only \( \land, \lor, \forall, \exists \).

**Theorem 2** (Lyndon homomorphism theorem, 1959). — A sentence \( \varphi \) is preserved under homomorphic images if and only if there is a positive sentence \( \psi \) which has exactly the same models as \( \varphi \).

The hard direction is "only if".

**Examples.** — The theories of groups, abelian groups, rings, and fields (if we allow the one element field) are preserved under homomorphic images because their axioms are positive. But the theory of integral domains is not preserved under homomorphic images. It has the axiom

\[
\forall x \forall y (x = 0 \lor y = 0 \lor \neg x \cdot y = 0),
\]

and this axiom cannot be replaced by a positive sentence.

A theory is complete if it is equal to \( \text{Th}(\mathcal{M}) \) for some \( \mathcal{M} \). Let us consider the number of (non-isomorphic) countable models of a complete theory \( T \).

**Examples.** — We have examples of complete theories with exactly one countable model (atomless Boolean algebras); \( \aleph_0 \) countable models (algebraically closed fields); \( 2^{\aleph_0} \) countable models (real closed fields); and \( n \) countable models for each \( n \geq 3 \) (due to Ehrenfeucht).

But the following surprising theorem is due to Vaught, 1959.

**Theorem 3.** — There is no complete theory which has exactly two countable models.

4. Elementary chains.

This construction was introduced by Tarski and Vaught, 1957.

\( \mathcal{A} \) and \( \mathcal{B} \) are said to be elementarily equivalent if \( \text{Th}(\mathcal{A}) = \text{Th}(\mathcal{B}) \), that is, they are models of exactly the same sentences.

\( \mathcal{A} \) is said to be a submodel of \( \mathcal{B} \), \( \mathcal{A} \subset \mathcal{B} \), if \( A \subset B \) and the operations, constants, and relations of \( \mathcal{A} \) are those of \( \mathcal{B} \) restricted to \( A \). \( \mathcal{A} \) is an elementary submodel of \( \mathcal{B} \), \( \mathcal{A} \subset \mathcal{B} \), if \( \mathcal{A} \subset \mathcal{B} \) and every sentence of \( \text{Diag}(\mathcal{A}) \) is true in \( \mathcal{B} \). A simple exercise: if \( \mathcal{A} \subset \mathcal{B} \) then \( \mathcal{A} \) and \( \mathcal{B} \) are elementarily equivalent.

**Example.** — Tarski, 1948 has shown that if \( \mathcal{B} \) is any real closed field and \( \mathcal{A} \) is a real closed subfield of \( \mathcal{B} \), then \( \mathcal{A} \subset \mathcal{B} \). Similarly for algebraically closed fields. Such theories are called model complete (Robinson, 1963).

An elementary chain is a sequence of models

\[
\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_\alpha, \ldots, \alpha < \gamma,
\]

where \( \gamma \) is an ordinal, such that

\[
\text{if } \alpha < \beta < \gamma \text{ then } \mathcal{A}_\alpha \subset \mathcal{A}_\beta.
\]

The union of an elementary chain is the model \( \mathcal{A} = U_\alpha < \gamma \mathcal{A}_\alpha \) such that \( A = U_\alpha < \gamma A_\alpha \) and each \( \mathcal{A}_\beta \) is a submodel of \( \mathcal{A} \).
**Theorem 4** (Tarski-Vaught, 1957). — Let $\mathfrak{A}_\alpha, \alpha < \gamma$, be an elementary chain. Then each $\mathfrak{A}_\beta$ is an elementary submodel of $U_{\alpha < \gamma} \mathfrak{A}_\alpha$.

A typical application of this construction is the following Löwenheim-Skolem-Tarski type result for pairs of cardinals. For this theorem we assume that the language contains a one-placed relation symbol $U$.

By a model of type $(\mathcal{N}_\alpha, \mathcal{N}_\beta)$ we mean a model $\mathfrak{U}$ such that $\mathcal{A}$ has cardinal $\mathcal{N}_\alpha$ and $U^{\mathfrak{U}}$ has cardinal $\mathcal{N}_\beta$.

**Theorem 5** (Vaught, 1962). — Suppose a theory $T$ has a model of type $(\mathcal{N}_\alpha, \mathcal{N}_\beta)$ where $\mathcal{N}_\alpha > \mathcal{N}_\beta$. Then $T$ has a model of type $(\mathcal{N}_1, \mathcal{N}_0)$.

The model $\mathfrak{U}$ of type $(\mathcal{N}_1, \mathcal{N}_0)$ is constructed as the union of an elementary chain $\mathfrak{A}_\alpha$, $\alpha < \omega_1$, of $\mathcal{N}_1$ countable models such that all the sets $U^{\mathfrak{A}_\alpha}$ are the same.

Many results in model theory depend on the Generalized Continuum Hypothesis (GCH), which states that for all infinite cardinals $\mathcal{N}_\alpha$, $2^{\mathcal{N}_\alpha} = \mathcal{N}_{\alpha+1}$. One such result is the following.

**Theorem 6** (Chang, 1965) (GCH). — Suppose a theory $T$ has a model of type $(\mathcal{N}_1, \mathcal{N}_0)$. Then for every $\mathcal{N}_\alpha$, $T$ has a model of type $(\mathcal{N}_{\alpha+2}, \mathcal{N}_{\alpha+1})$.

The proof uses an elementary chain of length $\mathcal{N}_{\alpha+2}$ of models of cardinality $\mathcal{N}_{\alpha+1}$.

**Example** (GCH). — Let $\mathfrak{U}$ be the model

$$\mathfrak{U} = < R, +, \cdot, 0, 1, <, Z >.$$  

where $R$ is the set of real numbers and $Z$ is the set of integers. $\mathfrak{U}$ is a model of type $(\mathcal{N}_1, \mathcal{N}_0)$. By Chang's theorem, $\text{Th}(\mathfrak{U})$ also has a model of type $(\mathcal{N}_2, \mathcal{N}_1)$. But $\text{Th}(\mathfrak{U})$ cannot have a model of type $(\mathcal{N}_2, \mathcal{N}_0)$.

## 5. Ultraproducts.

This construction was introduced by Skolem, 1934 to get a non-standard model of arithmetic and in its present general form it is due to L. Los, 1955.

Let $I$ be a non-empty set and let $\mathfrak{U}_i, i \in I$, be models for $L$. An ultrafilter over $I$ is a set $D$ of subsets of $I$ such that $D$ is closed under finite intersections, any superset of a member of $D$ is in $D$, and for all $X \subseteq I$, exactly one of the sets $X, I - X$ belongs to $D$. A statement $P(i)$ is said to hold almost everywhere (D) if the set of $i \in I$ for which $P(i)$ holds is in $D$.

Now consider the Cartesian product $\prod_{i \in I} A_i$. For $f, g \in \prod_{i \in I} A_i$ we write

$$f =_D g \iff f(i) = g(i) \text{ a.e.} \quad (D)$$

Then $=_D$ is an equivalence relation on $\prod_{i \in I} A_i$. Let $f_D$, be the equivalence class of $f$, and $\Pi_D A_i$ the set of all equivalence classes.

The ultraproduct $\Pi_D \mathfrak{A}_i$ is a model with the set of elements $\Pi_D A_i$. The relation $<$ on this model is defined by

$$f_D < g_D \iff f(i) < g_i \text{ a.e.} \quad (D).$$
The operation $+$ is defined so that
$$f_D + g_D = h_D \iff f(i) + g(i) = h(i) \text{ a.e. (D)}. $$

The fundamental result about ultraproducts is the following.

**Theorem 7** (Löb, 1955). — For each sentence $\phi$, $\omega$ holds in the ultraproduct $\Pi_D \mathcal{A}_i$ if and only if $\phi$ holds in $\mathcal{A}_i$ almost everywhere (D).

Ultraproducts can be used to give us another proof of the Compactness Theorem. Many applications of the Compactness Theorem can be done more neatly using ultraproducts directly.

**Example.** Suppose all the models $\mathcal{A}_i$ are fields, and form the complete direct product ring $\Pi_{\mathcal{A}_i}$. It turns out that the set of ultraproducts $\Pi_D \mathcal{A}_i$ is exactly the same as the set of quotient fields $\Pi_{\mathcal{A}_i} / J$ of the ring $\Pi_{\mathcal{A}_i}$ modulo a maximal ideal $J$ (The fields $\Pi_{\mathcal{A}_i} / J$ were studied by Hewitt, 1948; see Gillman-Jerison, 1960).

Suppose all the models $\mathcal{A}_i$ are the same model $\mathcal{A}$. Then the ultraproduct $\Pi_D \mathcal{A}$ is called an ultrapower of $\mathcal{A}$. By the theorem of Löb, $\mathcal{A}$ is elementarily equivalent to each ultrapower $\Pi_D \mathcal{A}$.

**Example** (non-standard analysis, A. Robinson, 1966). — Let $\mathcal{A}$ be the model
$$\mathcal{A} = \langle R, +, \ldots, 0, 1, <, \ldots \rangle$$
where $R$ is the set of real numbers, and the three dots stand for a list of all the $2^{\aleph_0}$ operations, constants, and relations on $R$. Let $D$ be an ultrafilter over the set $\omega = \{0, 1, 2, \ldots \}$ which contains no finite set. Then the ultrapower $\Pi_D \mathcal{A}$ is a non-Archimedean real closed field; for instance, $\langle 1, 1/2, 1/3, 1/4, 1/5, \ldots \rangle_D$ is a positive infinitesimal and $\langle 1, 2, 3, \ldots \rangle_D$ is positive infinite. Using the ultrapower $\Pi_D \mathcal{A}$, the whole subject of analysis can be based on infinitesimals in the style of Leibniz. For example, consider any real function $f$ and real numbers $c$ and $L$. Then $\lim_{x \rightarrow c} f(x) = L$ if and only if for every $b$ in $\Pi_D A$ which is infinitely close but not equal to $c$, $f(b)$ is infinitely close to $L$.

Ultrapowers can also be used to give purely algebraic characterizations of model-theoretic notions such as elementary equivalence.

**Theorem 8** (Isomorphism theorem). — Two models $\mathcal{A}$, $\mathcal{B}$ are elementarily equivalent if and only if there is an ultrafilter $D$ such that $\Pi_D \mathcal{A}$ and $\Pi_D \mathcal{B}$ are isomorphic.

This theorem was proved by Keisler, 1963, using the GCH, and was proved without the GCH by Shelah, 1971.

Among the important tools in model theory are the saturated models; they are used in theorems 6 and 8 above. The ultraproduct is one way of constructing such models. Let $\aleph_\alpha$ be an uncountable cardinal. $\mathcal{A}$ is $\aleph_\alpha$-saturated iff for every set $\Phi$ of fewer than $\aleph_\alpha$ formulas $\varphi(x)$ in the diagram language of $\mathcal{A}$, if for each $\varphi_1, \ldots, \varphi_\alpha \in \Phi$ the sentence
$$\exists x \ (\varphi_1(x) \land \ldots \land \varphi_\alpha(x))$$
is true in $\mathcal{A}$, then the infinitely long sentence
$$\exists x \land_{\varphi \in \Phi} \varphi(x)$$
is true in $\mathcal{A}$. 

146  H. J. KEISLER  G
THEOREM 9. — Let $I$ be a set of power $\aleph_a$. There is an ultrafilter $D$ over $I$ such that every ultraproduct $\prod_D \mathfrak{U}_i$ is $\aleph_{a+1}$-saturated.

The above result was proved under the GCH by Keisler, 1963, and without the GCH by Kunen, 1970. $\aleph_{a+1}$-saturated models were first constructed in another way by Morley-Vaught, 1962.

Example. — It turns out that a real closed field is $\aleph_a$-saturated if and only if its ordering is an $\eta_a$-set, that is, for any two subsets $X, Y$ of power $\aleph_a$ (perhaps empty), if $X < Y$ then there is an element $z$ such that $X < z < Y$.

There are a number of applications of saturated models to algebra. For example, they are the main tool in the proof by Ax and Kochen, 1965 of Artin's conjecture: for each positive integer $d$, the following holds for all but finitely many primes $p$. Every polynomial in the field $\mathbb{Q}_p$ of $p$-adic numbers, with degree $d$, more than $d^2$ variables, and zero constant term, has a non-trivial zero in $\mathbb{Q}_p$.

6. Indiscernibles.

Suppose we expand the language $L$ by adding $n$ new constant symbols $c_1, \ldots, c_n$, forming $L_n$. For each model $\mathfrak{U}$ for $L$ and each $n$-tuple $a_1, \ldots, a_n$ of elements of $\mathfrak{U}$, we obtain a model $(\mathfrak{U}, a_1, \ldots, a_n)$ for $L_n$. Consider a subset $X$ of $A$ and a linear ordering $<$ of $X$, which is not necessarily one of the relations of $\mathfrak{U}$. We say that $\langle X, < \rangle$ is a set of indiscernibles in $\mathfrak{U}$ if for any $n$ and any two increasing $n$-tuples

$$a_1 < \ldots < a_n, \quad b_1 < \ldots < b_n$$

from $\langle X, < \rangle$, the models $(\mathfrak{U}, a_1, \ldots, a_n)$ and $(\mathfrak{U}, b_1, \ldots, b_n)$ are elementarily equivalent. The basic result below shows that there always are models with indiscernibles.

THEOREM 10 (Ehrenfeucht-Mostowski, 1956). — Let $\mathfrak{T}$ have infinite models and let $\langle X, < \rangle$ be any linearly ordered set. Then there is a model $\mathfrak{U}$ of $\mathfrak{T}$ such that $\langle X, < \rangle$ is a set of indiscernibles in $\mathfrak{U}$.

The construction of the model $\mathfrak{U}$ uses the partition theorem of Ramsey.

Examples. — Let $\mathfrak{U}$ be a field and $\mathfrak{B}$ be the ring of polynomials over $\mathfrak{U}$ with the set $X$ of variables. Then for any linear ordering $<$ of $X$, $\langle X, < \rangle$ is a set of indiscernibles in $\mathfrak{B}$.

Let $\mathfrak{U}$ be a non-Archimedean real closed ordered field and let $X$ be a set of positive infinite elements such that if $x < y$ in $X$ then $x^n < y$, $n = 1, 2, \ldots$. Then $X$ with the natural order is a set of indiscernibles in $\mathfrak{U}$.

Indiscernibles are used to prove results such as the following (Two elements $a, b \in A$ have the same automorphism type if there is an automorphism of $A$ mapping $a$ to $b$).

THEOREM 11 (Ehrenfeucht-Mostowski, 1956). — If $\mathfrak{T}$ has an infinite model, then for every infinite cardinal $\aleph_a$, $\mathfrak{T}$ has a model of power $\aleph_a$ with only countably many automorphism types.

The following very deep results use both the method of indiscernibles and saturated models.

A theory $\mathfrak{T}$ is said to be $\aleph_a$-categorical if all models of $\mathfrak{T}$ of cardinal $\aleph_a$ are isomorphic.
Theorem 12 (Morley, 1965). — If \( T \) is \( \aleph_\alpha \)-categorical for some uncountable \( \aleph_\alpha \), then \( T \) is \( \aleph_\beta \)-categorical for every uncountable \( \aleph_\beta \).

Shelah, 1970, extended Theorem 12 to uncountable languages.

Theorem 13 (Baldwin-Lachlan, 1970). — If \( T \) is \( \aleph_1 \)-categorical, then either \( T \) is \( \aleph_0 \)-categorical or \( T \) has exactly \( \aleph_0 \) models of cardinal \( \aleph_0 \).

We mention one theorem at the opposite extreme from the above.

Theorem 14 (Shelah, 1970). — Suppose \( T \) has a model \( A \) such that for some formula \( \varphi(x, y) \) and some infinite set \( X \subset A \), the relation

\[
\{ \langle a, b \rangle \in X^2 : \forall \mathcal{U} \models \varphi(a, b) \}
\]

is a linear order. Then for every uncountable \( \aleph_\alpha \), \( T \) has \( 2^{\aleph_\alpha} \) non-isomorphic models of cardinal \( \aleph_\alpha \).

Example. — The theory of algebraically closed fields is \( \aleph_\alpha \)-categorical for every uncountable \( \aleph_\alpha \) and has \( \aleph_0 \) countable models. The theory of abelian groups with all elements of order two is \( \aleph_\alpha \)-categorical for every \( \aleph_\alpha \). The theory of real closed fields has \( 2^{\aleph_\alpha} \) models of each infinite cardinal \( \aleph_\alpha \). The theory of atomless Boolean algebras is \( \aleph_0 \)-categorical but has \( 2^{\aleph_\alpha} \) models of each uncountable cardinal \( \aleph_\alpha \).

7. Recent trends.

The model theory of first order logic contains a number of substantial results, but until recently only the compactness theorem has had many applications. This situation is changing and will change more as the subject becomes more widely known. One of the bottle-necks has been that most properties arising in mathematics cannot be expressed in first order logic. For this reason there is a strong move toward model theory for more powerful logics. In the last few years there have been exciting developments in the model theory of the infinitary logic \( L_{\omega_1\omega} \). This logic is like first order logic except that it allows the connectives \( \land \) and \( \lor \) to be applied to countable sets of formulas, that is, if \( \varphi_0, \varphi_1, \varphi_2, \ldots \) are formulas of \( L_{\omega_1\omega} \), then so are

\[
\varphi_0 \land \varphi_1 \land \varphi_2 \land \ldots, \quad \varphi_0 \lor \varphi_1 \lor \varphi_2 \lor \ldots
\]

The formulas may thus be countable in length.

Examples. — The sentence

\[
\forall x \ (x = 0 \lor x + x = 0 \lor x + x + x = 0 \lor \ldots)
\]

is true in an abelian group \( G \) if and only if \( G \) is a torsion group. The sentence

\[
\forall x \ (x < 1 \lor x < 1 + 1 \lor x < 1 + 1 + 1 \lor \ldots)
\]

is true in an ordered field if and only if it is Archimedean.

Both the Compactness Theorem and the LST Theorem in their original form are false for \( L_{\omega_1\omega} \). For the latter, note that every Archimedean ordered field has power \( \leq 2^{\aleph_0} \). Nevertheless, it turns out that all of the methods from first order model theory can be used in \( L_{\omega_1\omega} \). Many of the main results have been generalized to
$L_{\omega_1\omega}$, often in a more subtle form. For example, the LST Theorem takes the following form. The cardinal $\kappa$ is defined by the rule

\[
\kappa_0 = \aleph_0, \quad \kappa_{a+1} = 2^{\kappa_a},
\]

and $\kappa_\alpha = \Sigma_{\beta<\alpha} \kappa_\beta$ for limit ordinals $\alpha$.

**Theorem 15** (Morley, 1965). Let $\varphi$ be a sentence of $L_{\omega_1\omega}$. If $\varphi$ has a model of cardinal at least $\kappa_{\omega_1}$, then $\varphi$ has models of every infinite cardinal.

The proof is much deeper than the LST Theorem. It uses the partition calculus of Erdős and Rado, 1956, and also yields an analog of Theorem 10 on indiscernibles for $L_{\omega_1\omega}$.

Theorems 1 and 2 above were extended to $L_{\omega_1\omega_1}$ by Lopez-Escobar, 1965, Theorem 5 by Keisler, 1966, various forms of Theorem 12 by Choodnovsky, Keisler, and Shelah, 1969, and Theorem 14 by Shelah, 1970.

Another basic result is

**Theorem 16** (Scott, 1965). For every countable model $M$ there is a sentence $\varphi$ of $L_{\omega_1\omega}$ such that $M$ is a model of $\varphi$ and every countable model of $\varphi$ is isomorphic to $M$.

This result is analogous to Ulm's theorem for countable abelian torsion groups. In fact, $L_{\omega_1\omega}$ has been applied by Barwise and Eklof, 1970 to extend Ulm's theorem to arbitrary abelian torsion groups.

The model theory for $L_{\omega_1\omega}$ is greatly enriched by the use of recursion theory as a way to get a hold on infinitely long sentences (a suggestion of Kreisel). This has led to the Barwise Compactness Theorem (Barwise, 1969) which is the analog for $L_{\omega_1\omega}$ of the Compactness Theorem.

Another type of logic where model theory has had recent successes is logic with extra quantifiers, such as "there exist infinitely many" and "there exist uncountably many". For more information see the paper [12].

A major recent trend is the impact of set theory on model theory and vice versa. A number of problems have been shown to be consistent or independent using Cohen's forcing, notably by Silver. Moreover, forcing itself is being used as a technique for constructing models (see A. Robinson's lecture in this Congress). Other results have been proved on the basis of strong hypotheses such as the existence of a measurable cardinal (Rowbottom and Gaifman, 1964, Silver, 1966, Kunen, 1970), or the axiom of constructibility. For example, Jensen, 1970 has shown that if the axiom of constructibility holds then Chang's Theorem 6 above can be improved to:

If $T$ has a model of type $(\mathcal{N}_1, \mathcal{N}_0)$ then $T$ has a model of type $(\mathcal{N}_{\omega+1}, \mathcal{N}_\omega)$.

**BIBLIOGRAPHY**

The three new books [4], [7] and [11] taken together cover most facets of model theory and contain up to date references as of 1970. An extensive list of references to journal articles prior to 1963 can be found in [1]. We have decided that a list of books on model theory and related topics will be more useful at this time than another long list of references to the original articles in journals.

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