RECURRENCE IN OBJECTS OF FINITE TYPE

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My hope here in Nice is to draw attention to the work of S. C. Kleene [7] on recurrence in objects of finite type. In pursuit of that hope I will touch lightly on some related developments in generalized recursion theory. My Nicene creed is: Kleene's notion of recursive object of finite type and Gödel's notion of constructible set are of similar, but not of the same, substance. An Athanasian might see them as the same after reading Shoenfield [20] on hierarchies, but the Arian view is more balanced in the light of Moschovakis [11, 12] on hyperprojective sets.

I owe much to R. Gandy, T. Grilliot, and P. Hinman, who patiently explained to me the concept of recurrence in objects of finite type, and to G. Kreisel [8], who taught me that such things as "concepts" exist in the context of recursion theory.

An object of type 0 is a natural number. An object of type \( n > 0 \) is a total function whose arguments and values are of type \( < n \). \( U, V, \ldots \) denote objects of finite type. Kleene [7] introduced a transitive relation \( U \leq V \) (to be read \( U \) is recursive in \( V \)). If \( U \) and \( V \) are objects of type 1, then \( \leq \) coincides with Turing reducibility. For each finite type, 0 ambiguously denotes the function of that type which is everywhere equal to 0. If \( U \leq 0 \), then \( U \) is said to be recursive. If \( U \leq V \) and \( V \leq U \), then \( U \equiv V \) (to be read \( U \) and \( V \) have the same degree). \( X, Y, \ldots \) denote members of \( 2^\omega \) called reals, and \( F, G, H, \ldots \) denote total functions from the reals into the reals.

For each \( n > 0 \), \( 2^E \) is the characteristic function of equality for objects of type \( < n \). Thus \( 2^E(X, Y) = 0 \) if \( X = Y \), and \( = 1 \) otherwise. \( 2^E \) has the same degree as the Turing jump operator. A result of great internal beauty obtained by Kleene [7] is: the objects of type \( \leq 2 \) recursive in \( 2^E \) are just the hyperarithmetic ones. \( S^1_k \), the \( k \)-section of \( U \), is the set of all objects of type \( k \) recursive in \( U \). Kleene [7] asked: do there exist \( F \)'s such that (1) \( S^1_1 F \) consists of the arithmetic reals?, (2) \( S^1_1 F \) consists of the \( \Delta^1_1 \) reals? Recently Grilliot [5] answered (1) negatively by showing: if \( S^1_1 F \) is closed under the Turing jump, then \( 2^E \leq F \). (2) is answered affirmatively below.

Platek [13] calls a transitive set \( A \) admissible if \( A \) is closed under finitary set operations and all instances of the \( \Sigma_1 \) reflection and \( \Delta_0 \) comprehension axiom schemas are true in \( A \). A function \( f \) from \( A \) into \( A \) is called \( \Delta \)-recursive if the graph of \( f \) is a \( \Sigma_1 \) subset of \( A \). For every \( F \) it is possible to construe \( S^1_1 F \) as a countable transitive set \( AS^1_1 F \) by exploiting the standard encoding of hereditarily countable sets by reals. An immediate consequence of Shoenfield [20], Hinman [6], and Grilliot [5] is: \( AS^1_1 F \) is admissible if and only if \( 2^E \leq F \). It follows from Gandy's work [3] on selection operators that if \( 2^E \leq F \), then \( AS^1_1 F \) satisfies the \( \Sigma_1 \) dependent choice axiom schema.

THEOREM 1 [17]. — (i) and (ii) are equivalent.
(i) $A$ is a countable admissible set that satisfies the $\Sigma_1$ dependent choice axiom schema, and every member of $A$ is countable in $A$.

(ii) There exists an $F$ of type 2 such that $^2E \leq F$ and $A = AS_1F$.

For each ordinal $\alpha$ let $L_\alpha$ be the set of all constructible sets of constructible order $< \alpha$. Kripke [10] and Platek [13] call $\alpha$ an admissible ordinal if $L_\alpha$ satisfies the $\Sigma_1$ replacement axiom schema. A function is $\alpha$-recursive if its graph is a $\Sigma_1$ subset of $L_\alpha$. A set is $\alpha$-recursively enumerable if it is the range of an $\alpha$-recursive function. If $\alpha$ is admissible, then by Gödel there is an $\alpha$-recursive well-ordering of all the computations needed to compute all the $\alpha$-recursive functions. It follows that a great deal of classical recursion theory can be generalized from $\omega$ to $\alpha$. For example, it can be shown that the Friedberg-Muchnik solution of Post's problem holds for every admissible $\alpha$. The recursive functions of ordinals were first defined by Takeuti [22]. One of his early results restated in current terms says that every cardinal is an admissible ordinal. His proof is an application of Gödel's Skolem-Lowenheim principle for $L$: $\Sigma_1$ subsystems of $L$ are isomorphic to initial segments of $L$. Gödel's principle (with $L$ replaced by $L_\alpha$) is central to current work on admissible ordinals; it plays an unexpected part in the solution of Post's problem [19]. I say "unexpected" because the use of model-theoretic ideas in recursion theory was at one time a surprise to me. On the other hand Kreisel's approach [8, 9] to generalized recursion theory was based from the beginning on the model-theoretic notion of implicit invariant definability. Later Barwise showed by means of a compactness argument: if $A$ is a countable admissible set, then the implicitly invariantly definable functions from $A$ into $A$ are equivalent to the $A$-recursive functions (The equivalence fails for most uncountable $A$'s). I would like to recommend the joint paper [1] of Barwise, Gandy, and Moschovakis as a starting point for any one curious about the great variety of ideas now current in generalized recursion theory.

**Corollary 2.** — If $\alpha$ is a countable admissible ordinal, then there exists an $F$ of type 2 such that $L_\alpha \cap 2^\alpha = S_1F$, and such that for every $G$ of type 2 and of lower degree than $F$, $L_\alpha \cap 2^\alpha \neq S_1G$.

**Corollary 3.** — If $n > 0$, then there exists an $F$ of type 2 such that the reals recursively in $F$ are just the $\Delta^2_n$ reals.

**Theorem 4.** — If $U$ is of type $n$ and $^nE \leq U$, then $S_1 U = S_1 F$ for some $F$ of type 2.

The above four results are proved with the aid of Gödel's Skolem-Lowenheim principle for $L$, Cohen's forcing method, and Grilliot's hierarchies based on objects of finite type [4]. The next theorem combines forcing with the Friedberg-Muchnik priority method. Platek [13] calls $X$ $F$-recursive in $Y$ if $X \notin F$, $Y$. Two reals have the same $F$-degree if each is $F$-recursive in the other. Hinman calls a real $F$-recursively enumerable if it is the range of a partial function of type 1 recursive in $F$. A well-known result of Spector [21] can be extended to show: if $^2E \leq F$, then all non-$F$-recursive, $F$-recursively enumerable reals have the same $F$-degree. I say $X$ is $\Sigma_1$ in $Y$ over $AS_1F$ if $X$ is a $\Sigma_1$ subset of $AS_1F(Y)$, where $AS_1F(Y)$ is the result of adjoining $Y$ to $AS_1F$ and closing under $\Delta_0$ comprehension.

**Theorem 5.** — If $^2E \leq F$, then there exist two $F$-recursively enumerable reals such that neither is $\Delta_1$ in the other over $AS_1F$. 
Kleene [7] showed that the $^2E$-recursively enumerable reals were just the $\Pi^1_1$ reals. Theorem 5 for $F = ^2E$ was proved in [15].

The superjump is a fundamental object of type 3 introduced by Gandy [3]; it lifts $F$ to $F^1$. Let $\{ e \}^F(X)$ denote the value (possibly undefined) of the $e$-th partial function of type 2 recursive in $F$ for real argument $X$. The value of $F^1(e, X)$ is 0 if $\{ e \}^F(X)$ is defined and 1 otherwise. $^2E^1$ is the hyperjump and has the same degree as $E_1$, an object of type 2 associated with the Souslin operation and introduced by Tugué [23]. Gandy [3] showed: if $F \leq G$, then $F^1 \leq G^1$. Hinman has asked: is there a condition on $G$ that implies the existence of an $F$ such that $F^1 \equiv G$? Hinman’s question was inspired by Friedberg’s classic result [2]: if $J0 \leq X$, then there exists a $Y$ such that $JY \equiv X$, where 0 is the empty set and $J$ is the Turing jump.

**Theorem 6** [18]. — Assume the continuum hypothesis. Then there exists an $H$ such that $(G)(EF)[H \leq G \rightarrow F^1 \equiv G]$.

The $F$'s of Theorems 1 through 5 are constructed in countably many steps, but the $F$ of Theorem 6 is constructed in uncountably many steps. If the continuum hypothesis is dropped, then Theorem 6 can be approximated in the sense of Theorem 7. The continuum hypothesis is needed to make the approximations cohere with one another.

**Theorem 7.** — If $S^1 G$ is closed under hyperjump, then there exists an $F$ such that $S^1 F^1 = S^1 G$.

The next theorem is intended to suggest that the Tugué hierarchy for $S^1 E_1$ is similar to the Shoenfield hierarchy for $S^1 F^1$ whenever $^2E \leq F$; it was proved in [16] for the case of $F = ^2E$.

**Theorem 8.** — If $^2E \leq F$, then the $F$-degrees of $S^1 F^1$ have a minimal, but no least, upper bound.

Most of the results of this paper have the following form: a structure $B$ associated with some generalization of recursion theory is given; then an object $U$ of type $n$ is constructed such that the members of $B$ coincide with the objects of type $< n$ that are recursive in $U$. Since Kleene’s definition of relative recursiveness is inductive, it follows that $B$ can be defined by an induction based on $U$. If enough results of the above form can be found, it may be possible (as Kreisel has suggested) to prove theorems about structures occurring in generalizations of recursion theory by thinking of them as having been built up by inductive definitions based on objects of finite (or higher) type. Among the means to that end would be various sharpenings of Theorem 4. The superjump $^3S$ is an object of type 3 of lower degree than $^3E$, but an application of Corollary 2 above to Platek [14] provides an $F$ of type 2 such that $S^1^3S = S^1 F$. So it seems likely that the hypothesis “$^nE \leq U$” of Theorem 4 can be replaced by something of wider scope. Theorem 4 can be extended from 1-sections to $k$-sections as follows. For each $n$ there is a $V$ of type $n$ such that for all $U$ of type $n$: if $V \leq U$ and $k < n$, then $S_k U = S_k W$ for some $W$ of type $k + 1$. It is possible that $^nE$ may suffice for $V$, but at the moment I need a $V$ whose degree appears to be higher than $^nE$ save when $k = 1$ (Added in proof: if Gödel’s axiom of constructibility holds, then $V = ^nE$).
BIBLIOGRAPHY

[17] —. — The 1-section of a type \( n \) object, to appear.