For the past thirty or forty years mathematicians have increasingly been attracted to problems in the calculus of variations in higher dimensions and codimensions. However, prior to 1960 (with one or two notable exceptions) there was relatively little fundamental progress in the calculus of variations in higher dimensions and codimensions and essentially no progress on the so called parametric problems (those which are of an essentially geometric character). Beginning about ten years ago, however, (in particular with the work of De Giorgi, Federer, Fleming, and Reifenberg) new ideas began to be introduced into the subject with surprising success in these higher dimensions and codimensions. Indeed, in these higher dimensions and codimensions, the calculus of variations seems to have passed from a classical period in its development into a modern era. Many of these new methods and ideas are included in the collection of mathematical results known as geometric measure theory (see, in particular, the treatise [Fl]). This article is intended as a very brief discussion of several problems in the calculus of variations as an indication of the perspective from which these problems are now being studied.

The parametric boundary value problem.

Parametric boundary value problems arise in the following way: Suppose $k$ and $n$ are positive integers with $k \leq n$ and one is given a reasonably nice function $F : R^n \times G^n_k \rightarrow R^+$ where $G^n_k$ denotes the Grassmann manifold of all unoriented $k$ plane directions in $R^n$ (which can be regarded as the space of all unoriented $k$ dimensional planes through the origin in $R^n$). If $S$ is a reasonably nice surface of dimension $k$ in $R^n$, one defines the integral $F(S)$ of $F$ over $S$ by setting $F(S) = \int_{x \in S} F(x, S(x)) \, dH_k(x)$ where $S(x)$ denotes the tangent $k$ plane direction to $S$ at $x$ and $H_k$ denotes $k$ dimensional Hausdorff measure on $R^n$. Hausdorff $k$ dimensional measure gives a precise meaning to the notion of $k$ dimensional area in $R^n$ and is the basic measure used in defining a theory of integration over $k$
dimensional surfaces in $R^n$ which may have singularities. The Hausdorff $k$ dimensional measure of a smooth $k$ dimensional submanifold of $R^n$ agrees with any other reasonable definition of the $k$ area of such a manifold. With this terminology, the problem can be stated:

**Problem. — Among all $k$ dimensional surfaces $S$ in $R^n$ having a prescribed boundary, is there one minimizing $F(S)$? And, if there is, how nice is it?**

To make this problem precise, there are, of course, several questions to be answered: (1) What is a surface?, (2) What is the boundary of a surface?, and (3) What are reasonable conditions to put on $F$? To see what is involved in answering these questions, one needs to study the phenomena which arise. Indeed, even for the case $F \equiv 1$ (i.e. the problem of minimizing $k$ dimensional area—often called Plateau's problem), examples [A3] show: (1) In order to solve the problem of least area—and really achieve the least area—one sometimes has to admit surfaces of infinite topological type into competition (even for two dimensional surfaces whose boundaries are piecewise smooth simple closed curves); (2) Complex algebraic varieties are surfaces of least oriented area, so that, in particular, at least all the singularities of complex algebraic varieties occur in solutions to Plateau's problem; (3) In some cases there are topological obstructions to surfaces of least area being free of singularities; (4) Sometimes surfaces of least area do not span their boundaries in the sense of algebraic topology; and (5) The realization of certain soap films as mathematical "minimal surfaces" requires that the boundary curves have positive thickness.

One approach to the study of variational problems in the generality suggested by the phenomena which arise is based on a correspondence between suitable surfaces and measures on appropriate spaces. Indeed, the natural setting for parametric (1) problems in the calculus of variations seems to be that in which surfaces are regarded as intrinsically part of $R^n$ (in particular as measures on spaces associated with $R^n$) rather than that in which surfaces are regarded as mappings from a fixed $k$ dimensional manifold, even though with this approach one is not able to use the traditional methods of functional analysis for showing the existence of solutions. The principal reasons for formulating the problem this way are indicated in [A3]. The most important measure theoretic surfaces are indicated in the following:

(1) **Rectifiable sets.** — A set $S \subset R^n$ is $k$ rectifiable if and only if $H_k(S) < \infty$ and $H_k([S \sim f(A)] \cup [f(A) \sim S]) = 0$ for some measurable set $A \subset R^k$ and some Lipschitzian function $f : R^k \rightarrow R^n$.

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(1) Traditionally one has considered surfaces as mappings $f$ from a fixed $k$ dimensional manifold $M$ into $R^n$ and attempted to minimize the integral of a suitable integrand. If the integrand depends only on $x \in M$, $f(x)$, and $Df(x)$ and if the integral of the integrand is independent of the parametrization of $M$ (as is the case for the area integrand, but is not the case for most "energy" integrands), the variational problem is said to be in parametric form. Problems in parametric form are precisely those problems for which the necessary integration can be performed over the image $f(M)$ in $R^n$. 
(2) Variation measures. — If \( S \) is \( k \) rectifiable, the variation measure \( \| S \| \) associated with \( S \) is given by the formula \( \| S \| = H_k \cap S \) (i.e. \( \| S \| (A) = H_k (S \cap A) \) for \( A \subset R^n \)). \( \| S \| \), of course, determines \( S \) \( H_k \) almost uniquely, but it is difficult to evaluate \( F_S \) from \( S \) alone.

(3) Integral varifolds. — If \( S \) is \( k \) rectifiable, the integral varifold \( | S | \) is given by the formula \( | S | = \varphi_* (\| S \|) \), where \( \varphi : R^n \to R^n \times G_k^n \), \( \varphi(x) = (x, S(x)) \) for \( \| S \| \) almost all \( x \) in \( R^n \). Note that \( \ psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi \psi 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boundary of class $j + 1$. Finally, the ellipticity of $F$ implies that the various Euler equations which arise are strongly elliptic systems of partial differential equations, and “in the small” the ellipticity of $F$ is equivalent to the ellipticity of these systems.

**Theorem [A2].** — *Let $B$ be a boundary, $G$ be a finitely generated abelian group, and $\sigma \in H_{k-1}(B; G)$. Suppose $F : \mathbb{R}^n \times G^k \to \mathbb{R}^+$ is an integrand of class $j \geq 3$ which is elliptic with respect to $G$ and which is bounded away from 0. Then there exists a surface $S$ such that $S$ spans $\sigma$, $F(S) \leq F(T)$ whenever $T$ is a surface which spans $\sigma$, and, except possibly for a compact singular set of zero $H_k$ measure, $S$ is a $k$ dimensional submanifold of $\mathbb{R}^n$ of class $j - 1$.*

A second class of variational problems.

So far we have been concerned with boundary value problems for a wide class of integrands. A second class of problems relates to the study of a particular integrand—the area integrand—under conditions where the surfaces in question are subject to distortions, constraints, or other influences as well as being permitted to have arbitrary topological type and essential singularities. Solutions to these problems usually are not minimal surfaces, nor surfaces minimal for any integrand. To be precise, we need some more terminology.

By a *varifold* we mean a Radon measure on $\mathbb{R}^n \times G^k$. By the area $W(V)$ of a varifold $V$ we mean $V(\mathbb{R}^n \times G^k)$. Each smooth diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ induces a map $f_\#$ of varifolds in a natural way characterized by requiring that $f_\# | S = | f(S)|$ whenever $S$ is $k$ rectifiable. By the *first variation* $\delta V$ of a varifold $V$ we mean the distribution on $\mathbb{R}^n$ of type $\mathbb{R}^n$ and order 1 given by $\delta V(g) = (d/dt) W(G_t # V)|_{t=0}$ (first variation of area) for each vectorfield $g : \mathbb{R}^n \to \mathbb{R}^n$ where $G_t : \mathbb{R}^n \to \mathbb{R}^n$ is the deformation given by $G_t(x) = x + tg(x)$ for $x \in \mathbb{R}^n$. For example, if $M$ is a smooth submanifold of $\mathbb{R}^n$, then $\delta |M(g)| = - \int_M g \cdot \vec{m} \cdot dH_k + \int_{\partial M} g \cdot \vec{n} \cdot dH_{k-1}$ where $\vec{m}$ is the mean curvature vectorfield on $M$, and $\vec{n}$ is the exterior normal vectorfield on $\partial M$ tangent to $M$. One says that the first variation $\delta V$ is a *measure* if and only if $\delta V$ is of order 0, i.e. there is a covector valued measure $\mu$ on $\mathbb{R}^n$ such that $\delta V(g) = \int g \, d\mu$ for each vectorfield $g$. Finally, by the $k$ density $\Theta^k(\|V\|, p)$ of a varifold $V$ at a point $p \in \mathbb{R}^n$ we mean $\lim_{r \to 0^+} r^{-k} V(\{x : |x - p| < r\} \times G^k)$.

A variety of physical and biological phenomena have mathematical representation by varifolds whose first variation distributions are measures; for example, spider webs (here the tension in each strand corresponds to the density of the corresponding varifold), soap bubbles as well as soap films, liquid-liquid interfaces in equilibrium, and partitioning surfaces of least weighted area (such as occur as interfaces in the stable states of free living cells). These phenomena remain representable by varifolds whose first variations are measures when subject to gravitational fields, wind pressures, etc. As a representative simple mathematical example we have the following:
PARTITIONING PROBLEM. — Let $m_i > 0$ for $i = 1, 2, \ldots, j$. Among all disjointed regions $A_1, A_2, \ldots, A_j$ in $\mathbb{R}^n$ such that $L_2(A_i) \geq m_i$ for each $i$, are there regions for which $H_{n-1}(\bigcup_i \partial A_i)$ attains a minimum value? (if so one can verify that $\delta \mid \bigcup_i \partial A_i \mid$ is a measure).

The varifold setting seems both a natural and a powerful way to study a variety of geometric and variational problems, including those suggested by the physical and biological phenomena above. The following results are suggestive of the present state of the theory.

**Theorem.** — If $V$ is a varifold and $\delta V$ is a measure, then $\{ p : \Theta^k(\|V\|, p) > \varepsilon \}$ is a $k$ rectifiable set for each $\varepsilon > 0$.

**Isoperimetric Inequality.** — If $V$ is a varifold, $W(V) < \infty$ and $\delta V$ is a measure, then $W(V)^{(k-1)/k} \inf \{ \Theta^k(\|V\|, p) : p \in \text{support} \|V\| \} \leq c \|\delta V\|$. For example, if $M$ is a minimal surface, $H_k(M) \leq c_1 H_{k-1}(\partial M)^{k/(k-1)}$. Here $c$ and $c_1$ are constants depending only on $n$.

By an integral varifold one means a varifold which can be represented $\sum_i |S_i|$ corresponding to rectifiable sets $\{S_i\}$.

**Theorem.** — The space of integral varifolds with locally uniformly bounded areas and first variations which are measures is compact in the weak topology.

**Theorem.** — On each compact $n$ dimensional Riemannian manifold without boundary there exists at least one nonzero $k$ dimensional integral varifold $V$ with $\delta V = 0$. $V$ is thus a $k$ dimensional "minimal surface". (The proof is by Morse Theory methods).

**Regularity Theorem [AL].** — If $S$ is a $k$ rectifiable set and $\delta |S|$ is integrable to the $k + \varepsilon$ power, then, except possibly for a compact singular set of zero $H_k$ measure, $S$ is a smooth $k$ dimensional submanifold of $\mathbb{R}^n$ with first derivatives which are locally Holder continuous with exponent $\varepsilon/(k + \varepsilon)$.

**Solution to the Partitioning Problem.** — The theory of integral currents guarantees a solution to the partitioning problem such that $\bigcup_i \partial A_i$ is $n-1$ rectifiable and $\delta |\bigcup_i \partial A_i|$ is bounded. The regularity theorem implies that $H_{n-1}$ almost everywhere $\bigcup_i \partial A_i$ is a smooth submanifold. The regular part of $\bigcup_i \partial A_i$ has locally constant mean curvatures, hence is an analytic manifold.

Estimates on singular sets.

Very little is known at the present time about the structure of the singular sets of solutions to general elliptic variational problems (except for their existence). However, for the area integrand there has been substantial progress. For example, we have the following two representative theorems which generalize immediately to manifolds.

**Theorem [F2].** — For every unoriented boundary $B \subset \mathbb{R}^n$ of dimension $k-1$ (any $k$), there exists an unoriented minimal surface (flat chain modulo 2) $S$ with $\partial S = B$ of least $k$ dimensional area. The interior singular set of $S$ has Hausdorff dimension at most $k - 2$. The regular part of $S$ is a real analytic submanifold of $\mathbb{R}^n$.
Theorem [F2]. — For every oriented boundary \( B \subset R^n \) of dimension \( n - 2 \) there exists an oriented minimal surface (integral current) \( S \) with \( \partial S = B \) of least \( n - 1 \) dimensional area (counting multiplicities). The interior singular set of \( S \) has Hausdorff dimension at most \( n - 8 \). In particular, there are no interior singularities if \( n \leq 7 \). The regular part of \( S \) is a real analytic submanifold of \( R^n \).

Examples show that both of these results are the best possible (at least in terms of Hausdorff dimension).

Example. — The unoriented 2 dimensional surface \( S = \{x : x_3 = x_4 = 0 \text{ and } x_1^2 + x_2^2 \leq 1\} \cup \{x : x_1 = x_2 = 0 \text{ and } x_3^2 + x_4^2 \leq 1\} \subset R^4 \) is of least area among all unoriented 2 dimensional surfaces having boundary \( B = \{x : x_3 = x_4 = 0 \text{ and } x_1^2 + x_2^2 = 1\} \cup \{x : x_1 = x_2 = 0 \text{ and } x_3^2 + x_4^2 = 1\} \). The origin 0 is the singular set of \( S \).

Example [BDG]. — Let \( S \) be the 7 dimensional oriented cone \( 0(S^3 \times S^3) \) over \( S^3 \times S^3 \subset R^7 \times R^7 = R^{14} \). Then \( S \) has less 7 dimensional area than any other oriented hypersurface \( T \) in \( R^8 \) with \( \partial T = S^3 \times S^3 \). The origin 0 is the singular set of \( S \).

The results above for oriented minimal hypersurfaces are intimately connected with the possibility of extending Bernstein's theorem (that a globally defined nonparametric minimal hypersurface must be a hyperplane) to higher dimensions.

Theorem [F1 5.4.18] [BDG]. — If \( n = 2, 3, \ldots, 7 \) and if the graph of \( f : R^n \to R \) is a minimal hypersurface in \( R^{n+1} \), then the graph of \( f \) is a hyperplane. On the other hand, for each \( n \geq 8 \) there exist functions \( g : R^n \to R \) with graphs which are minimal hypersurfaces but which are not hyperplanes.

REFERENCES

For a complete set of references to geometric measure theory one should consult the Bibliography of [F1].


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