DIFFERENTIABILITY THEOREMS FOR NON-LINEAR ELLIPTIC EQUATIONS*

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1. Introduction.

I shall discuss equations of the form

\[
\int_G \sum_{i=1}^N \sum_{|\alpha| \leq m_i} \phi_{\alpha}^i(x) A_{\alpha}^i(x, Dz(x)) \, dx = 0 \quad \text{for all } \xi \in C_0^\infty(G)
\]

where \( C_0^\infty(G) \) denotes the set of functions of class \( C^\infty \) with compact support in \( G \), \( x = (x_1, \ldots, x_N) \), \( \xi = (\xi_1, \ldots, \xi^N) \), and \( z = (z_1, \ldots, z^N) \) are vector functions, \( \alpha = (\alpha_1, \ldots, \alpha_N) \) denotes a multi-index in which each \( \alpha_i \) is a non-negative integer, \( |\alpha| = \alpha_1 + \cdots + \alpha_N \), and

\[
\phi_{\alpha}^i(x) \quad \text{and} \quad D^{\alpha} \phi(x) \quad \text{stand for } \frac{\partial^{|\alpha|} \phi}{(\partial x_1^{\alpha_1}) \cdots (\partial x_N^{\alpha_N})},
\]

and \( Dz \) stands for all the derivatives \( D^{\alpha} z^i \) for \( i = 1, \ldots, N \) and \( 0 \leq |\alpha| \leq m_i \) (of course if \( |\alpha| = 0 \), \( D^{\alpha} z^i = z^i \)).

Equations of the form (1) were discussed in my paper “Partial regularity theorems for elliptic systems” which appeared in the January 1968 issue of the Journal of Mathematics and Mechanics [17] where it was assumed that the \( A_{\alpha}^i \) are of class \( C^2 \) (\( 0 < \mu \leq 1 \)) in their arguments and satisfy

\[
|A(x, p)|, |A_{\alpha}^i(x, p)| \leq MV^{k-1} ; |A_p|, |A_{p\alpha}|, |A_{pp}| \leq MV^{k-2} ;
\]

\[
(2) \quad \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} A_{\alpha \beta}^i (x, p) \pi_{\alpha}^i \pi_{\beta}^j \geq m^* V^{k-2} |\pi|^2 , m^* > 0 ,
\]

\( k \geq 2 \), \( V^2 = 1 + \sum_{i=1}^N \sum_{|\alpha|=m_i} (\alpha_{i \alpha})^2 \quad (p = \{p_{i \alpha}^j\}, 0 \leq |\alpha| \leq m_i) \)

where

\[
(3) \quad |A_p|^2 \quad \text{means} \quad \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} (A_{i \alpha \beta})^2 \quad \text{etc}.
\]

I and a student have obtained similar results under somewhat more general hypotheses on the \( A_{\alpha}^i \) (see below).

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It is to be noticed that if the $A_i^a$ and $z^i$ are sufficiently differentiable, the terms of (1) involving the derivatives of $z^i$ may be integrated by parts so that (1) becomes

\[(1') \int_G \sum_i z^i \sum_{|a| \leq m_i} (-1)^{|a|} D^a A_i^a [x, Dz(x)] \, dx = 0, \text{ for all } \xi \in C^\infty_c(G).\]

However, the equations (1) make sense if the $z^i$ are merely in the Sobolev spaces $H^m_k$ (see below) in which case we say that $z$ is a \textit{weak solution} of the equations \{(\xi) = 0 in (l').

It is easy to see that the left side of (1) is just the so-called \textit{first variation} of an integral of the form

\[I(z) = \int_G f[x, Dz(x)] \, dx\]

if we set

\[A_i^a(x, \rho) = f_{p \alpha}^i (x, \rho).\]

Thus a discussion of the regularity of the solutions of (1) includes that of the extremals of multiple integral problems in the calculus of variations. Of course the equations (1) are more general since the $A_i^a$'s for a variational problem would satisfy (5) and hence

\[A_{ip}^a = A_{p i}^a\]

which we do not assume. The inequality in the middle of (2) just involves the symmetric part of the $A_p^a$ matrix. If $A$ satisfies (5), this inequality implies that $f$ is convex in the $p$-variables.

In the case where all the $m_i = 1$ and (5) holds, (1) becomes

\[(1'') I_1(z, \xi; G) = \int_G (\xi_{,a} f_{p \beta}^i + \xi^i f_{s i}) \, dx\]

where repeated indices are summed ($i$ from 1 to $N$, $a$ from 1 to $v$) and $\xi_{,a}$ now means $\partial \xi^i/\partial x^a$, $\alpha = 1, \ldots, v$. If $z, \xi, \text{ and } f \in C^1(G)$,

\[(7) \quad I_1(z, \xi; G) = \phi'(0) \quad \text{where } \phi(\lambda) = I(z + \lambda \xi; G)\]

and $I(z, G)$ is given by (4) which reduces in this case to

\[(4') \quad I(z, G) = \int_G f(x, z, \nabla z) \, dx, \quad \nabla z = \{z_{,i}^i\}.\]

In case we wish to discuss the minimizing character of a critical solution, it is desirable to discuss the \textit{second variation} defined by

\[(8) \quad \phi''(0) = I_2(z, \xi; G) = \int_G \left( f_{p \alpha \beta}^i \xi_{,a}^i \xi_{,a}^j + 2f_{p a i j} \xi_{,a}^i \xi_{,a}^j + f_{s d} \xi^i \xi^j \right) \, dx\]
in case $f$ is of class $C^2$. If $I_2(z, \xi; G) \geq 0$ for every $\xi \in C^1(G)$ which vanishes on $\partial G$, it follows that (see [16], p. 11)

$$ f_{\alpha \beta}^l \lambda_\alpha \lambda_\beta \xi^l \xi^l \geq 0 \forall \lambda, \xi. $$

In case $\nu = 1$ or $N = 1$ and (9) holds for all $(x, z, p)$ it follows that $f$ is convex in the $p$'s for each $(x, z)$. However, if $\nu > 1$ and $N > 1$, this is no longer true.

A variational problem for which the inequality holds in (9) for all $\lambda \neq 0$ and $\xi \neq 0$ is called regular. In this case the Euler equations

$$ \frac{\partial}{\partial x^\alpha} f^l_{\alpha \beta} = f^l_{z \beta} \text{ or } f^l_{\alpha \beta \gamma} z^l_{\alpha \beta} + f^l_{p^\alpha \beta} z^l_{\alpha} + f^l_{p^\alpha \beta} \gamma^l = f^l_{x^l}, \quad l = 1, \ldots, N $$

(which are satisfied weakly by any critical solution) are strongly elliptic in the sense of Nirenberg ([22], [16], § 6.5). The condition in the middle of (2) corresponds in the present case to assuming that the quadratic form

$$ f_{\alpha \beta}^l \gamma^l_{\alpha \beta} $$

is positive definite which implies that $f$ be convex in the $p$'s for each $(x, z)$. Many of our results have this requirement and therefore do not include regularity results for the most general regular variational problems. The number $k$ in (2) is seen in our case to be the degree of $f$ at $\infty$ as a function of $p$, since our assumptions imply that

$$ f(x, z, p) \geq f(x, z, 0) + f^l_{p^l}(x, z, 0) p^l $$

$$ + |p|^2 \int_0^1 [1 + |z|^2 + t^2 |p|^2]^{(k-2)/2} dt $$

When discussing a variational problem of this sort, it is customary to adjoin the condition

$$ m |p|^k - K \leq f(x, z, p) \leq M |p|^k + K, \quad 0 < m \leq M. $$

In the case where the $A^x_l$ satisfy (5), the existence of a solution can be proved by using the so-called direct methods of the calculus of variations developed by Tonelli and others to show the existence of a minimizing function; if $f$ satisfies the conditions above, the minimizing vector satisfies (1). The idea of the direct methods is to show (i) that the integral to be minimized is lower semicontinuous with respect to some kind of convergence, (ii) that it is bounded below in some class of "admissible functions" and (iii) that there is a "minimizing sequence," i.e., a sequence for which the integral tends to its infimum, which converges in the sense required to some admissible function.

For the one dimensional problems ($\nu = 1$) with all the $m_i = 1$, Tonelli found it expedient to allow absolutely continuous functions as admissible and to use uniform convergence. This comes about roughly as follows: Suppose that

$$ f(x, z, p) \geq m |p|^r - K, \quad r > 1, \quad m > 0 $$
(which is not unreasonable since \( f \) is convex in \( p \)) or satisfies (13). Then

\[
\int_a^b |z_n'(x)|^r \, dx \leq L, \quad n = 1, 2, \ldots
\]  

in any minimizing sequence \( \{z_n\} \). From (15), one sees from the Hölder inequality that any minimizing sequence is equi-continuous. Moreover if (a subsequence of) \( z_n \) converges uniformly to some function \( z \) on \([a, b]\), then \( z \) is absolutely continuous and

\[
\int_a^b |z'(x)|^r \, dx \leq \liminf_{n \to \infty} \int_a^b |z_n'(x)|^r \, dx.
\]

Then \( z \) would be minimizing if the integral were lower-semicontinuous with respect to that type of convergence.

Unfortunately the equicontinuity of minimizing sequences is not implied by (14) \((|p|^2 = \sum_{i,a} (p_i')^2)\) unless \( r > \nu \). This fact led me in the Fall of 1937 to introduce function spaces, now to be identified with the so-called “Sobolev spaces”, in order to carry through the program for cases where \( \nu > 1 \). We assume that the reader is familiar with these spaces. We denote by \( H^m_p(G) \) those “functions” (i.e., distributions) whose (distribution) derivatives up to the \( m \)-th order are in \( L_p \) on \( G \). These functions are defined and discussed in the author’s book ([16], Chapter 3). We recall here that \( fH^1_p(G) \) is the closure in \( H^m_p(G) \) of the set \( C_0\). Using these functions, we may state a lower semicontinuity theorem and an existence theorem as follows (see [16], 22-24):

**Theorem (Lower semicontinuity).** — Suppose \( f = f(x,z,p) \) and the \( f_{p_\nu} \) are continuous with \( f(x,z,p) \geq 0 \) for all \((x,z,p)\), suppose \( f \) is convex in \( p \) for each \((x,z)\), and suppose \( z_n \rightharpoonup z \) (tends weakly to) \( z \) in \( H^1_p(D) \) for each \( D \subset G \). Then

\[
I(z, G) \leq \liminf_{n \to \infty} I(z_n, G).
\]

**Theorem (Existence).** — Suppose \( f \) satisfies the conditions of the preceding theorem as well as (14), suppose \( G \) is bounded, \( z^* \in H^1_p(G) \), and \( I(z^*, G) < \infty \). Then there exists a \( z \in H^1_p(G) \) such that \( z - z^* \in H^1_p(G) \) and \( z \) minimizes \( I(z, G) \) among all such functions.

More general existence theorems have been proved, of course. And recently existence theorems have been proved for equations of the general type (1); these involved the theory of monotone operators and its extensions developed by Visik, Minty, Browder, Leray, Lions and others (see [16], § 5.12). But these yield only the conclusion that each \( z^t \) of a solution belongs to \( H^m_p(G) \).

The principal purpose of this paper is to present some results which state that solutions (possibly weak) of certain elliptic systems have additional differentiability properties. One of the first results of this sort was that due to S. Bernstein in 1904, who proved that any solution of class \( C^{(3)}(G) \) of an analytic
non-linear elliptic equation of the second order in one unknown function and two independent variables \((N = 1, \nu = 2)\) is analytic. His proof was long and many others (including himself) gave simpler proofs and extended his results. These analyticity results have been extended to very general elliptic systems and include results concerning analytic extensions across an analytic boundary of solutions satisfying general regular boundary conditions (in the sense of Agmon, Douglas, and Nirenberg [1]) ; for references see the author’s presidential address [18] p. 688, Bull. Amer. Math. Soc., 75 (1969), p. 688.

A somewhat different series of generalizations of Bernstein’s result was begun by L. Lichtenstein [10] when he showed in 1912 that a solution of class \(C^2(G)\), \(G \subset \mathbb{R}^2\), of an analytic variational problem \((\nu = 2, N = 1)\) is of class \(C''\) and hence analytic. This result was extended in 1929 by E. Hopf [8] to the case where the solution was required only to be of class \(C^1_\mu(G)\) for some \(\mu, 0 < \mu < 1\).

The author [11] extended this result further in January 1938 to the case where the solution was required merely to satisfy a Lipschitz condition. Somewhat earlier, Haar [7] showed the existence and uniqueness of a Lipschitz solution of any variational problem in which \(\nu = 2, N = 1, f = f(p), \partial G\) is strictly convex, and the given boundary values satisfy a three point condition (i.e., \(\exists \text{ an } M \ni \text{ any plane intersecting the boundary curve in three points has slope } \leq M\)) ; this solution is analytic by the author’s result.

During the year 1937-38, the author proved in the case where \(f\) satisfies conditions a little more general than the corresponding conditions in (2) with \(k = \nu = 2, N\) arbitrary, \(m, \mu, \alpha, \beta = 1\), that the solution vector \(z \in C^n(G)\) if \(f \in C^n(G)\) and \(n \geq 3\). These results were presented in the seminar of Marston Morse at the Institute for Advanced Study during the Spring of 1938 ; the notes were written by H. Busemann and are in the library at the Institute under Busemann’s name. They were also presented in an invited address before the American Mathematical Society in the Fall of 1939 [12]. A greatly simplified account of this work is to be found in the author’s Pisa lectures [13], especially Chapter 4.

In the work above the author used a certain “Dirichlet growth” principle which did not generalize to more than two variables. No progress was made on this problem until the famous results of De Giorgi [3] and Nash [19] who proved independently that any solution \(u\) in \(H^1_2(G)\) of an equation of the form

\[
(16) \quad \int_G \xi_{\alpha a} a^{a\beta}(x) u_{,\beta} \, dx = 0 \quad \xi \in H^1_2(G)
\]

in which the \(a^{a\beta}\) are bounded and measurable and satisfy

\[
(17) \quad m |\lambda|^2 \leq a^{a\beta}(x) \lambda_{a \alpha} \lambda_{\beta} \leq M |\lambda|^2, \quad 0 < m < M, \quad x \in G,
\]

are Hölder continuous on interior domains. De Giorgi used this to show that if \(z\) is a solution \(\in H^1_2(G)\) of a problem in which \(N = 1, k = 2, \nu\) arbitrary, \(f = f(p)\), then \(z \in C^m_\mu(G)\) (analytic, \(C^n\)) if \(f\) is. During the year 1959-60, the author and a student E. R. Buley [14], [15] and concurrently O. A. Ladyzenskaya and N. N. Ural’tseva [9] established corresponding regularity results for equations of the form \((1)\) in which \(N = 1\) and all the \(m_i = 1\) but \(\nu\) and \(k\) are arbitrary. The
were required to satisfy conditions closely resembling (2); the exact conditions for the work of this author and his student are to be found in the author's book [16], p. 33.

For several years there were no results for the cases where \( N > 1 \) except some results on higher order systems in \( \nu = 2 \) variables by J. Nečas [20], [21]. Meanwhile, Reifenberg [23], [24], [25], Almgren [2], and others have shown the existence of solutions of the parametric problem in higher dimensions which solutions were each the union of smooth open manifolds with a locally compact set of (\( \nu \)-dimensional) measure 0. Their methods are entirely different from those involved in the non-parametric problem. However, the author was able to adapt some of Almgren's theorems to prove the following theorem [17]:

**Theorem.** — Suppose the \( A_i^a \) are of class \( C^2 \) and satisfy (2), suppose each \( z^i \in H^{m/2}_k(G) \), and suppose \( z \) is a solution of (1). Then each \( z^i \in C^{m+2} \mu(D) \) where \( D = G - Z \) and \( Z \) is locally compact, and of measure zero.

In view of the discovery by E. Giusti and M. Miranda [5] of an analytic variational problem which has the unique extremal

\[
u^i = |x|^{-1} x^i, \quad x \in B(0,1)
\]

the theorem above is of some interest. This example and the theorem above focus interest on the properties of the set \( Z \). In a recent paper, Giusti [4] has shown under assumptions related to ours that \( Z \) has Hausdorff \((\nu - 1)\) dimensional measure zero. In another recent paper [6], Giusti and Miranda have proved the following theorem:

**Theorem.** — Suppose the \( a 's \) are bounded and uniformly continuous for all \((x, u)\) and satisfy

\[
a^a_{ij}(x, u) \prod^i \pi^j \geq |\pi|^2
\]

for all \((x, u, \pi)\). Suppose that \( u \in H^1_p(\Lambda) \) for some \( p \geq 2 \) and all domains \( \Delta \subseteq G \) and suppose \( u \) is a solution of

\[
\int_G \xi^i a^a_{ij}(x, u) u^j \ dx = 0 \ \forall \xi \in \text{Lip}_c(G)
\]

Then \( u \) is Hölder continuous on each compact subset of a domain \( D = G - Z \) where \( Z \) is locally compact with Hausdorff \((\nu - p)\)-dimensional measure zero. \( \text{Lip}_c G \) is all Lipschitz functions with compact support in \( G \).

**BIBLIOGRAPHY**


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