THE FLOW OF FLUIDS IN POROUS MEDIA

by Jim DOUGLAS Jr

1. Introduction.

This lecture will outline the developments over the past two years of Todd Dupont and the author on problems of flow of fluids in porous media arising from petroleum reservoir engineering. Two specific problems will be discussed and a third mentioned. The first is the single-phase flow of a non-ideal gas. This flow can be described by a nonlinear parabolic equation. The second is the two-phase flow of water and oil. If the fluids are assumed incompressible (this is the more difficult situation mathematically) and immiscible, a degenerate nonlinear parabolic system in two dependent variables results. The time derivative of only one appears explicitly. The third problem comes from a so-called “beta factor” description of three-phase (oil, gas, water) flow. Again a very strongly nonlinear parabolic system is generated.

Galerkin methods are used both to treat the theoretical questions of existence, uniqueness, and continuous dependence and the practical problem of approximating the solutions for the single-phase and two-phase problems. Approximate solution by Galerkin procedures is discussed for the three-phase problem, but the other issues remain untouched in this case.


Let us consider the flow of gas in a porous medium [3, 4]. For simplicity, consider the domain to be horizontal and radially symmetric so that a single space variable can be used to describe the geometry. Then the flow can be described by the parabolic equation

\[
\frac{\partial}{\partial t} \left( 2 \pi r \phi \rho \right) = \frac{\partial}{\partial r} \left( 2 \pi r \frac{k \rho}{\mu} \frac{\partial p}{\partial r} \right),
\]

where

\[
0 < r_{\text{min}} \leq r < r_{\text{max}} < \infty,
\]

and \( \phi = \phi_1(r) \phi_2(p) \) is the porosity of the medium, \( \rho \) is the density of the gas, \( k = k(r) \) is the permeability of the medium, \( \mu = \mu(p) \) is the gas viscosity, and \( p \) is the gas pressure. Assume that

\[
\zeta = \zeta(p) = \frac{p}{\rho}
\]
is the equation of state (assuming the temperature fixed). It is convenient to make the following change of variables:

\[ q = q(p) = \int_0^p \frac{\tau \, d\tau}{\xi(r) \mu(r)} . \]

Let

\[ c(r) = 2\pi r \phi_1(r) , \quad a(r) = 2\pi r k(r) \]

and

\[ f(q) = \frac{p \phi_2(p)}{\xi(p)} , \quad f'(q) = \mu \left\{ \phi_2 \left( \frac{1}{p} - \frac{\xi'}{\xi} \right) + \phi_2 \right\} . \]

Then the differential equation reduces to the form

\[ c(r) f'(q) \frac{\partial q}{\partial t} = \frac{\partial}{\partial r} \left( a(r) \frac{\partial q}{\partial r} \right) . \]

Initial values for \( p \) and, hence, for \( q \) need to be specified. Either \( p \) or the flow, \( a \partial q/\partial r \), should be given for \( 0 < t \leq T \) at the boundaries.

Existence, uniqueness, and continuous dependence of the solution on the data can be treated (for arbitrary dimension) quite easily by exactly the same methods as will be discussed in the next section. Let us consider the problem of approximating the solution by a Galerkin method. Assume that flows are specified and that \( \mathcal{K} \subset H^1(r_{\min}, r_{\max}) \). If \( \{ v_1, \ldots, v_N \} \) is a basis for \( \mathcal{K} \) and

\[ q(r, t) = \sum_{j=1}^N \eta_j(t) v_j(r) \]

is the approximate solution, the Galerkin method for determining \( \eta(t) \) is given by the system

\[ C(\eta(t)) \frac{d\eta(t)}{dt} + A(\eta(t)) = \gamma(t) , \]

where

\[ C(\eta) = \left( \int_{r_{\min}}^{r_{\max}} c(r) f' \left( \sum_k \eta_k v_k \right) v_i \, dr \right) , \]

\[ A = \left( \int_{r_{\min}}^{r_{\max}} a(r) \frac{dy}{dr} \frac{dy}{dr} \right) , \]

\[ \gamma_t = a \frac{\partial q}{\partial r} v_i \bigg|_{r_{\min}}^{r_{\max}} . \]

Note that \( A \) is independent of time. The specification of initial values translates into a specification of \( \eta(0) \). It can be proved [3] that for any reasonable choice of a family of subspaces \( \mathcal{K} \) converging to \( H^1 \), the solution \( \tilde{q} \) converges to \( q \).
under very reasonable assumptions. Moreover, the estimates hold with constants independent of \( \mathcal{M} \). Similar estimates are valid when (2.9) is solved by a Crank-Nicolson differencing in time or by various predictor-corrector versions of the Crank-Nicolson relation. The proofs are independent of the dimension of the domain.

A practical implementation of the Galerkin approximation has been made using an Hermite quintic basis modified to allow jump discontinuities in the coefficients \( c(r) \) and \( a(r) \) \cite{4}. The technique has proved to be most satisfactory.

3. Two-Phase Flow.

Let water and oil flow simultaneously in a porous medium. Consider the fluids to be in capillary equilibrium and to be incompressible and immiscible. Then this flow can be described by the degenerate parabolic system \cite{1, 5, 6}

\[
\begin{align*}
  c(u)_t &= \nabla \cdot (a \nabla u) + \nabla \cdot (b \nabla v) + f, \\
  0 &= \nabla \cdot (b \nabla u) + \nabla \cdot (a \nabla v) + g, \quad x \in \Omega, \quad 0 < t \leq T,
\end{align*}
\]

where

\[
  c = c(x, u), \quad a = a(x, u), \quad b = b(x, u), \quad f = f(x, u, \nabla u, \nabla v),
\]

and

\[
g = g(x, u, \nabla u, \nabla v).
\]

The variable \( u \) represents the capillary pressure and \( v \) the average of the two phase pressures. Assume that the coefficients are smooth in \( u \) and \( v \) and bounded and measurable in \( x \). Also, assume that

\[
a(x, u) > \alpha > 0, \quad |b(x, u)| \leq (1 - \epsilon) a(x, u).
\]

While it is possible to assign several types of boundary conditions, the choice arising most naturally is the following:

\[
\begin{align*}
  a \frac{\partial u}{\partial n} + b \frac{\partial v}{\partial n} &= -\psi, \quad x \in \partial \Omega^+, \\
  b \frac{\partial u}{\partial n} + a \frac{\partial v}{\partial n} &= \psi, \quad x \in \partial \Omega^-,
\end{align*}
\]

and

\[
\frac{\partial u}{\partial n} = 0, \quad a \frac{\partial v}{\partial n} = \varphi, \quad x \in \partial \Omega^-, \quad x \in \partial \Omega^-,
\]

where \( \partial \Omega^+ \) is the portion of \( \partial \Omega \) through which injection takes place and \( \partial \Omega^- \) the production portion. Consistency with incompressibility requires that

\[
\int_{\partial \Omega^+} \psi \, d\sigma + \int_{\partial \Omega^-} \varphi \, d\sigma + \int_\Omega g \, dx = 0.
\]
We also assume the normalization

\[ \int_\Omega \nu \, dx = 0. \]

Initially, \( u \), but not \( \nu \), must be specified.

The weak form of the above system is the following:

\[
\begin{align*}
\langle c(u)_t, z \rangle_\Omega &+ \langle a(u) \nabla u, \nabla z \rangle_\Omega + \langle b(u) \nabla \nu, \nabla z \rangle_\Omega \\
&= -\langle \psi, z \rangle_{\partial \Omega}^+ + \langle \frac{b}{a}(u) \varphi, z \rangle_{\partial \Omega}^- + \langle f, z \rangle_\Omega.
\end{align*}
\]

(3.7)

\[
\begin{align*}
\langle b(u) \nabla u, \nabla z \rangle_\Omega &+ \langle a(u) \nabla \nu, \nabla z \rangle \\
&= \langle \psi, z \rangle_{\partial \Omega}^+ + \langle \varphi, z \rangle_{\partial \Omega}^- + \langle g, z \rangle_\Omega,
\end{align*}
\]

where

\[ z \in H^1(\Omega), \ 0 < t \leq T, \]

(3.8)

\[ \langle \alpha, \beta \rangle_X = \int_X \alpha \beta \, d\xi, \quad d\xi \text{ being Lebesgue measure}. \]

The Galerkin method can be used to demonstrate the existence of a solution of (3.7) such that \( u \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), u_0 \in L^2(0, T; H^1(\Omega)'), \) and \( \nu \in L^2(0, T; H^1(\Omega)). \) A uniqueness result can be proved in the neighborhood of a smooth solution; i.e., if \( u \in C^1(\Omega \times [0, T]) \) and \((u, \nu)\) is a solution of (3.7) having \( u(0) = u_0 \), then for data in a neighborhood of \((u_0, \psi, \varphi)\) the weak solution depends continuously on the data.

The practical approximation of smooth solutions of (3.7) can also be accomplished by Galerkin methods. In the continuous time case, the error estimate [3] bounds

(3.9) \[ \| u - U \|_{L^2(0, T; L^2(\Omega))}^2 + \| u - U \|_{L^2(0, T; H^1(\Omega))}^2 + \| \nu - V \|_{L^2(0, T; H^1(\Omega))}^2, \]

\((U, V)\) being the approximate solution, in terms of the ability to approximate \((u, \nu)\) within the Galerkin subspace. Convergence proofs for the differenced-in-time versions of the Galerkin process have been obtained only under the simplifying assumption that \( c(x, u) = c_i(x)u \). The Crank-Nicolson equation can be shown to be second order correct in \( \Delta t \) and the backward equation in which the coefficients and the boundary conditions are evaluated at the old time level is first order correct. Predictor-corrector methods can also be employed. The nonlinear boundary conditions apparently need to be handled implicitly; also, two passes through the corrector seem to be required to maintain second order accuracy in \( \Delta t \). This second pass results from the degeneracy of the differential system, since it is not required for a non-degenerate system [2, 3].

Earlier (and not totally unblemished) versions of these results can be found in [1, 5].
4. Three-Phase Flow.

The 'beta factor' description of the flow of oil, gas, and water in a porous medium leads to a nonlinear parabolic system of three equations. Some of the coefficients depend discontinuously on the solution, and, in general, the system fails to fall into any of the commonly considered classes of parabolic systems. Dupont, H.H. Rachford, Jr., and the author have succeeded in devising an efficient Galerkin method for its approximate solution (at least as measured by the computer), but since we have no proofs the problem will not be treated further here.

REFERENCES


University of Chicago
Dept. of Mathematics,
5734 University Avenue
Chicago,
Illinois 60637 (USA)