E 4 - THÉORIE DU CONTROLE OPTIMAL

A CONSTRUCTIVE APPROACH
TO THE MAXIMUM PRINCIPE

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In control theory, the Maximum Principle has for the most part been a theory of necessary conditions that a postulated optimal solution must satisfy. There is a weakness here inherent in proving properties of an empty set, and Young [10] for instance cites a paradox of Perron in illustration. In partial resolution, McShane [11] and Young [10] (among others) prove existence of "relaxed" controls (also known as "chattering" controls, based on the "generalized curves" of Young), and then deduce an appropriately modified version of the Maximum Principle. The present approach goes one step further and is totally constructive. We develop a computational scheme which at the same time yields the Maximum Principle for a constructed limiting solution.

To be specific (and to stay within the space limitation) we shall consider a particular class of problems, not the most general.

Minimize

\[
\int_0^T g(t; x(t); u(t)) \, dt
\]

where \( T \) is fixed and finite, subject to :

\begin{align*}
(2) & \quad x(t) = f(t; x(t); u(t)) \quad \text{a.e.} \\
(3) & \quad x(0) = x_1, x(T) = x_2 \\
(4) & \quad \phi(t; x(t); u(t)) = 0 \quad \text{a.e.}
\end{align*}

The controls are Lebesgue measurable, and subject to additional constraints \( C \). We assume that there is at least one such control including (2) thru (4), with finite (1), so that the infimum, denoted \( g(0) \), is less than plus infinity.

Our approach is to replace this by an approximating non-dynamic problem. For each \( \epsilon > 0 \), minimize :

\[
\frac{1}{2 \epsilon} \int_0^T \| \dot{x}(t) - f(t; x(t); u(t)) \|^2 \, dt + \frac{1}{2 \epsilon} \int_0^T \| \phi(t; x(t); u(t)) \|^2 \, dt + \int_0^T g(t; x(t); u(t)) \, dt
\]

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over the class of controls $u(t)$, Lebesgue measurable and subject to $C$, and over the class of absolutely continuous functions $x(t)$ satisfying the end-conditions (3) and

$$\int_0^T (||x(t) - f(t ; x(t) ; u(t))||^2 + || \phi (t ; x(t) ; u(t)) ||^2) \, dt \leq M < \infty$$

(the last condition can be eliminated in $\inf g(t ; x ; u) > - \infty$). Note that because $x(t)$ is not required to satisfy the dynamic equations, we can incorporate additional "phase-plane" constraints, such as inequality constraints for example. Let $u_n(t)$, $x_n(t)$ be a minimizing sequence for (5). Let

$$\delta(e) = \lim \inf \frac{1}{2} \int_0^T (||x_n(t) - f(t ; x_n(t) ; u_n(t))||^2 + \frac{1}{2} || \phi (t ; x_n(t); u_n(t)) ||^2$$

and let \(d(e)\) denote the supremum of $\delta(e)$ over all such minimizing sequences, and correspondingly $g(e)$ the infimum of $G(e)$. Let $h(e)$ denote the infimum of (5), so that

$$h(e) = d(e)/e + g(e) \leq g(0)$$

We have then the basic estimate on how well the epsilon problem approximates the original problem. Assuming merely that $g(.)$, $\phi(.)$, $f(.)$ are say continuous, we have :

**Theorem.** — Suppose $g(e_0)$ is finite for some $e_0$. Then $g(e)$ is finite for every $e$ less than $e_0$, and $g(e)$ is monotone non-decreasing as $e$ decreases, and similarly $d(e)$ is monotone non-increasing. Moreover $h(e)$ is monotone and is differentiable omitting a countable number of points with

$$h'(e) = -d(e)/e^2 ,$$

and

$$g'(e) + d'(e) = 0 \text{ a.e.}$$

Also :

$$d(e)/e \leq g(0+) - g(e) \rightarrow 0$$

To proceed further, we need to be more specific. Thus we assume that the functions $f(.)$, $g(.)$, $\phi(.)$ are $C^1$ in $x$. Further, since ours is a constructive approach with existence included, we specify the simplest blanket conditions :

$$u(t) \in U \text{ compact a.e.}$$

and

$$[x, f(t ; x ; u)] = 0 [1 + ||x||^2 ]$$

[Alternately we can assume (ad hoc) only the minimal needed properties that these conditions imply, as in McShane [11] for example. We forego this gene-
rality in the interest of simplicity, especially since it is not an intrinsic limitation of our approach]. We may then modify (6) as:

$$\int_0^T ||x(t)||^2 dt < m < \infty$$

Note that in this case $g(e)$ in (7) is always finite.

Our computational procedure for minimizing (5) is to pick any “admissible” state function $x(t)$ (i.e., absolutely continuous with given end-conditions, and satisfying (10)). Then we choose a control that minimizes (5). The important point here is the simple one that this can be done by minimizing the integrand in (5). And thereby hangs the Maximum Principle. Let $\mathfrak{H}$ denote the class of regular probability measures on (the Lebesgue subsets of) $U$, and let $C(t;x)$ denote the compact convex set of points $x$:

$$x = \int_U f(t;x;u) \, d\mu, \int_U \phi(t;x;u) \, d\mu, \int_U g(t;x;u) \, d\mu$$

as $d\mu$ ranges over $\mathfrak{H}$. This is of course the closed convex hull of the set

$$\{f(t;x;u), \phi(t;x;u), g(t;x;u), \quad u \in U\}$$

It will be convenient to use the notation from now on:

$$\overline{f}(t;x;\mu) = \int_U f(t;x;u) \, d\mu$$

and similarly for the other functions. For each point $x$ in $C(t;x)$ let

$$r(e;t;y;x) = \frac{1}{2e} (||y - \overline{f}(t;x;\mu)||^2 + ||\phi(t;x;\mu) + \overline{g}(t;x;\mu)||)$$

and let

$$(11) \quad \overline{m}(e;t;y;x) = \inf_{x \in C(t;x)} r(e;t;y;x)$$

To obtain the Maximum Principle, let us note that the infimum in (11) is attained, and moreover letting

$$\overline{m}(e;t;y;x) = r(e;t;y;x_0)$$

we have, since $r(\ldots)$ is simply a quadratic functional on $C(t;x)$, that

$$\frac{d}{d\theta} r(e;t;y;x_0 + \theta(x - x_0))_{\theta=0} \geq 0$$

This differentiation yields:

$$\left[\Psi, \overline{f}(t;x;\mu_0)\right] + [\phi_0, \overline{\phi}(t;x;\mu_0)] - \overline{g}(t;x;\mu_0) =$$

$$\max \left[\Psi, \overline{f}(t;x;\mu)\right] + [\phi_0, \overline{\phi}(t;x;\mu)] - \overline{g}(t;x;\mu)$$

where

$$x_0 = \overline{f}(t;x;\mu_0), \quad \overline{\phi}(t;x;\mu_0), \quad \overline{g}(t;x;\mu_0); \quad \Psi = (y - \overline{f}(t;x;\mu_0))/\epsilon; \quad \phi_0 = \overline{\phi}(t;x;\mu_0)/\epsilon$$
and is recognized as the prototype of the Maximum Principle. Next let us note that if \( x_n(t), u_n(t) \) is a minimizing sequence for (5), then we may take a subsequence of \( x_n(t) \) to converge uniformly to \( x_e(t) \) say, while there is a relaxed control \( d\mu_e(t;u) \) such that

\[
\int_0^T dt \int_U k(t;u) \, d\mu_e(t;u) = \lim \int_0^T k(t;u_n(t)) \, dt
\]

for every continuous \( k(t;u) \). Moreover

\[
h(e) = \int_0^T m(t;x_e(t);x_e(t))
\]

\( h(e) \) being also the infimum in the class of relaxed controls.

Next let

\[
\Psi(e;t) = (\dot{x}_e(t) - \bar{f}(t;x_e(t);\mu_e(t)))/e
\]

\[
\phi_e(t) = \bar{\phi}(t;x_e(t);\mu_e(t))/e
\]

Then a first variation yields:

\[
(13) \quad \dot{\Psi}(e;t) + \bar{f}_1(t;x_e(t);\mu_e(t)) \Psi(e;t) - \bar{\phi}_1(t;\mu_e(t)) \phi_e(t) - 1 \frac{g(t;x_e(t);\mu_e(t))}{e} = 0
\]

where the subscripts denote gradient with respect to \( x \). An obvious substitution in (13) yields the approximate Maximum Principle:

\[
(14) \quad [\Psi(e;t) + \bar{f}(t;x_e(t);\mu_e(t))] + [\phi_e(t), \bar{\phi}(t;x_e(t);\mu_e(t))]
\]

\[-g(t;x_e(t);\mu_e(t)) = \text{Max}_{\mu} \{[\Psi(e;t) + \bar{f}(t;x(t);\mu)] + [\phi_e(t), \bar{\phi}(t;x(t);\mu)] - g(t;x_e(t);\mu)\}
\]

Finally let \( e \) go to zero. Then taking a suitable subsequence, \( x_e(t) \) converges uniformly to \( x_0(t) \) say, and \( d\mu_e(t;u) \) in the weak-star sense to \( d\mu_0(t;u) \) say, such that

\[
g(0+) = \int_0^T -\bar{g}(t;x_0(t);\mu_0(t)) \, dt
\]

\[
\dot{x}_0(t) = \bar{f}(t;x_0(t);\mu_0(t)) \text{ a.e.}
\]

\[
\bar{\phi}(t;x_0(t);\mu_0(t)) = 0 \text{ a.e.}
\]

Also \( g(0+) \) is the infimum for the control problem in the class of relaxed controls. The question now is that of taking limits in (13) and (14). We note that we need only the limits in the weak sense in \( L_2 \). We have no problems if for some sequence \( e_n \) going to zero, and some \( t_0 \),

\[
||\Psi(e_n;t_0)||^2 + \int_0^T ||\phi_{e_n}(t)||^2 \, dt < \infty
\]

Suppose then that
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\[
\liminf_{\epsilon \to 0} \||\epsilon(0)\||^2 + \int_0^T ||\phi(e(t))||^2 \, dt = +\infty
\]

If

\[
\liminf \int_0^T ||\phi(e(t))||^2 \, dt < \infty
\]

then we divide thru in (14) by

\[
k_n = ||\Psi(e_n;0)||
\]

and working with a subsequence, we obtain in the limit:

\[
[\Psi(t, f(t; x_0(t); \mu_0(t))) = \max [\Psi(t, f(t; x_0(t); \mu)] 
\]

\[
\Psi(t) + f(t; x_0(t); \mu_0(t)) = \Psi(t) = 0; \Psi(0) \neq 0
\]

Otherwise we divide by \(k_n\) where

\[
k_n^2 = ||\Psi(e_n;0)||^2 + \int_0^T ||\phi(e_n(t))||^2 \, dt
\]

In this case there is the possibility that the weak limits:

\[
\lim \Psi(e_n;0)/k_n = 0 = \lim \phi(e_n(t))/k_n
\]

are both zero. To avoid this, additional conditions involving derivatives of \(\phi(t; x; u)\) with respect to \(u\) can be imposed, for example those in [12]. The key consideration is whether \(d(e)/e^2\) is finite as \(e\) goes to zero and has a computational significance, as (8) shows. If the solution to the problem (5) is unique, then \(h(e)\) is actually absolutely continuous in \(e > 0\). The references listed may be consulted for more detail as well as application to other classes of problems.

REFERENCES


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