A General Framework for Simple $\Delta^0_2$ and $\Sigma^0_1$
Priority Arguments

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In this paper an abstraction of simple priority arguments is presented in terms of what are called priorcomeager sets. To illustrate the versatility of the method, a number of different applications are described in §§ 4 and 5. Among them, for example, are the existence of minimal $\Delta^0_2$ degrees and the nonexistence of minimal $\Sigma^0_1$ degrees.

Priorcomeager sets first appeared in our unpublished lecture notes [11], where separate but similar frameworks for $\Delta^0_2$ and $\Sigma^0_1$ arguments were presented. Subsequently, in [12], we have described a framework for the $\Delta^0_2$ theory in terms of what are called priorie games, connecting these with priorcomeager sets and deducing some additional corollaries. The present common framework for the $\Delta^0_2$ and $\Sigma^0_1$ theories is a considerable improvement on these earlier rather inelegant versions. It possesses the additional merit of dealing with some simple priority arguments in the $\Sigma^0_1$ theory below a fixed nonzero $\Sigma^0_1$ degree; these previously needed separate treatment.

Finally, it should be mentioned that Lachlan [2] has presented some elegant ideas for a framework which is restricted to the $\Sigma^0_1$ theory but deals with two harder theorems in that theory: the existence of minimal pairs of $\Sigma^0_1$ degrees and the density of the $\Sigma^0_1$ degrees.

1. Preliminaries. The reader is referred to the bibliography, in particular to the earlier papers of the author, for most standard notation and terminology. However, the following are sufficiently nonstandard but basic to our presentation that they merit repetition. If $T$ is a tree then

$$T \land \tau = \{ \sigma : \sigma \in T \& \tau \leq \sigma \}$$
\[ \mathcal{N}(T) = \{ B : B \in 2^\mathbb{N} \& B = \lim X \text{ for some branch } X \text{ of } T \}. \]

If \( T \) is a basic tree of the form \( S \wedge \tau \) then \( \mathcal{N}(\tau) \) will be used instead; in fact \( \tau \) will then replace \( T \) throughout. A tree system is, for our purposes here, a set of \( 2^\mathbb{N} \) trees containing \( S \) and closed under basic subtrees, i.e., if \( T \in C \) and \( \tau \in T \) then \( T \wedge \tau \in C \).

We follow [13] in writing \( 0 \) for the system of all basic trees and 1 for the system of all \( 2^\mathbb{N} \) trees; note that \( 0 \subseteq C \subseteq 1 \) for all systems \( C \) as used here. (Different notation was used in [12].) It will help to motivate the present paper if we recall the definition of a \((C, d)\)-comeager set, introduced and used in [13] for perfect systems \( C \). A C-probe is an operator \( Q : C \to C \) such that \( T \in Q(T) \) for all \( T \in C \).

\[ \mathcal{N}(Q_e(T)) \subseteq \mathcal{A}_e. \]

\( 2^\mathbb{N} \) is \((C, d)\)-comeager if \( \mathcal{A} \equiv \bigcap \mathcal{A}_e \) for some \((C, d)\)-dense sequence \( (\mathcal{A}_e) \). The \((C, d)\)-comeager sets are easily seen to be closed under supersets and finite intersections (even some infinite intersections). The existence theorem for \((C, d)\)-comeager sets is trivial and just a generalised form of Baire’s theorem. It asserts that if \( \mathcal{A} \) is \((C, d)\)-comeager then \( \mathcal{A} \) contains an element of degree \( \leq d \). This abstracts the usual genericity and diagonal arguments. Our purpose below is to present a suitable generalisation which abstracts priority arguments of the simplest kind.

2. Some triples \((C, \leq, d)\). Here \( C \) is a tree system (not necessarily perfect). \( \leq \) is a recursive binary relation over \( S \) (most conveniently assumed to extend the relation \( \leq_1 \) defined below) and \( d \) is some degree. The relations used in the most obvious applications are \( \leq, \leq_1, \leq_0, \leq_0^* \) and \( \leq_? \) all defined below. Although some results apply to all \( C \) we shall only specifically refer to \( 1 \) and 0. Also the usual values for \( d \) are 0 and \( 0^{(1)} \):

- \( \sigma \leq \tau \iff \sigma \equiv \tau \) is an extension of \( \sigma \).
- \( \sigma \leq_1 \tau \iff \sigma \equiv \tau \& |\tau| \leq |\sigma| + 1. \)
- \( \sigma \leq_0 \tau \iff \forall x (\sigma(x) = 0 \Rightarrow \tau(x) = 0) \& |\sigma| \leq |\tau|. \)
- \( \sigma \leq_0 \tau \iff \sigma \leq_1 \tau \& |\tau| \leq |\sigma| + 1. \)
- \( \sigma \leq_? \tau \iff \sigma \equiv_0 \tau \& (n(\sigma, \tau)) \leq n(\sigma, \tau), \)

where \( n(\sigma, \tau) = \min x (\sigma(x) \neq \tau(x)) \) and \( a \) is a 1-1 recursive function: \( N \to N. \)

\[ \sigma \leq_? \tau \iff \sigma \equiv_? \tau \& |\tau| \leq |\sigma| + 1. \]

2.1. DEFINITION. A \((C, \leq, d)\)-sequence is a sequence \( T_0, T_1, \cdots \) of elements of \( C \) which is of degree \( \leq d \) (in the sense that the sequence of indices of \( T_0, T_1, \cdots \) can be enumerated by a function of degree \( \leq d \)) and such that \( \mu(T_0) \equiv \mu(T_1) \equiv \cdots \).

If \( d \) is irrelevant we shall just refer to \((C, \leq)\)-sequences. Our interest lies in the elements of \( 2^\mathbb{N} \) to which \( \mu(T_0) \equiv \mu(T_1) \equiv \cdots \) may converge, where a sequence \( T_0, T_1, \cdots \) is said to converge to \( B \in 2^\mathbb{N} \) if for all \( \sigma < B \) there is a \( K \) such that \( \sigma \leq \tau_k \) for all \( k \geq K \). The purpose of the binary relations listed above will be clearer once we have noted that

(i) if a \((C, \leq, 0^{(1)})\)-sequence converges to \( B \in 2^\mathbb{N} \) then \( B \in \mathcal{D}_0 \), i.e., \( B \) represents a \( \mathcal{D}_0 \) set,

(ii) if a \((C, \leq, 0)\)-sequence converges to \( B \in 2^\mathbb{N} \) then \( B \in \mathcal{D}_1 \),

(iii) if a \((C, \leq, 0)\)-sequence converges to \( B \in 2^\mathbb{N} \) then \( B \in \mathcal{D}_0 \) and \( B \) is of degree
\( \leq \alpha \) where \( \alpha \) is the degree of the range of \( a \).

(i) is obvious. To see (ii) just observe that if \( \tau_k = \mu(T_k) \) for all \( k \), where \( T_0, T_1, \ldots \) is the sequence, then, for all \( x \),

\[
B(x) = 0 \iff \exists k(\tau_k(x) = 0).
\]

To see (iii) for a similar sequence just observe that, for all \( x \),

\[
B(x) = 0 \iff \exists k(\tau_k(x) = 0 \& |\tau_k| \leq M(x)),
\]

where \( M(x) = \max m(a(m) \leq x) \), then remember that \(|\tau_k|\) is monotonic.

Finally, the following definitions are important in the sequel. Let \( C^{<\omega} \) be the set of all nonempty finite sequences of elements of \( C \).

2.2. **Definition.** For any operator \( Q: C^{<\omega} \to C \) we define its *trace* \( \hat{\Omega} \) by setting

\[
\hat{\Omega}(T_0, \ldots, T_n) = Q(T_0, \ldots, T^n) \text{ for the largest } m < n \text{ for which this is defined if such an } m \text{ exists,}
\]

\[
T^0 \text{ otherwise.}
\]

2.3. **Definition.** A \((C, \leq, d)\)-probe is an operator \( Q: C^{<\omega} \to C \) satisfying

- **O**: \( \mu(T^n) \leq \mu(Q(T_0, \ldots, T^n)) \),
- **P**: \( T_0 \geq Q(T_0, \ldots, T^n) \),
- **R**: if \( \hat{\Omega}(T_0, \ldots, T^n) \cap \mu(T^n) \) is trivial (i.e., a singleton) but \( T_0 \cap \mu(T^n) \) is not trivial then \( \hat{\Omega}(T_0, \ldots, T^n) \) is defined.

*Note.* The third condition \( R \) (for *remedial*) is redundant for perfect systems \( C \) such as \( \emptyset \). The purpose of \( O \) in producing \((C, \leq, d)\)-sequences should be obvious enough. The purpose of \( P \) is more subtle, but it may be regarded simply as the appropriate generalisation of the definition of \( C \)-probe.

3. **(C, \leq, d)-priorcomeager sets.** We begin with the most crucial concept in our framework.

3.1. **Definition.** Let \( Q \) be a \((C, \leq, d)\)-probe. A \((C, \leq)\)-sequence \( T_0, T_1, \ldots \) is \( \Omega \)-prioric if, for all \( n \),

- (a) if \( \hat{\Omega}(T_0, \ldots, T^n) \) is defined then it is \( T^{n+1} \),
- (b) \( \hat{\Omega}(T_0, \ldots, T^n) \supseteq T^n \).

*Note.* It is immediate that \( T^0 \supseteq T^n \) for all \( n \), using 2.3P.

3.2. **Definition.** A sequence \((\mathcal{A}_e)\) of subsets of \( 2^N \) is \((C, \leq, d)\)-dense if there is a sequence \((\Omega_e)\) of \((C, \leq)\)-probes which is uniformly of degree \( \leq d \) and such that, for each \( e \) and \( \Omega_e\)-prioric \((C, \leq)\)-

- (I) \( \lim_n \Omega_e(T_0, \ldots, T^n) \) exists (= \( T^N \) say),
- (II) \( \mathcal{A}(T^N) \subseteq \mathcal{A}_e \).

3.3. **Definition.** A set \( \mathcal{A} \subseteq 2^N \) is \((C, \leq, d)\)-priorcomeager if \( \mathcal{A} \supseteq \bigcap \mathcal{A}_e \) for some \((C, \leq, d)\)-dense sequence \((\mathcal{A}_e)\).

*Note.* By 3.1(b) we have \( T^N \supseteq T^n \) for all \( n \geq N \) in 3.2, so that if \( \mu(T^0), \mu(T^1), \ldots \) converges to \( B \in 2^N \) then \( B \in \mathcal{A}_e \). This suggests two weaker notions described simultaneously below and of importance in the applications.

3.4. **Definition.** A sequence \((\mathcal{A}_e)\) of subsets of \( 2^N \) is weakly \((C, \leq, d)\)-dense (under \( d \)) if there is a sequence \((\Omega_e)\) as in 3.2 such that, for each \( e \) and \( \Omega_e\)-prioric \((C, \leq)\)-


sequence $T^0, T^1, \ldots \left((C, \preceq, d)\right)$-sequence $T^0, T^1, \ldots$,

(I) as in 3.2,

(II) if $\mu(T^0), \mu(T^1), \ldots$ converges to $B \in 2^N$ then $B \in \mathcal{A}$.

3.5. **Definition.** A set $\mathcal{A} \subseteq 2^N$ is weakly $(C, \preceq, d)$-priorcomeager (under $d$) if $\mathcal{A} \supseteq \bigcap \mathcal{A}_{e}$ for some $(\mathcal{A}_e)$ which is weakly $(C, \preceq, d)$-dense (under $d$). Each type of $(C, \preceq, d)$-priorcomeager set is easily seen to be closed under supersets and finite intersections (even some infinite intersections).

An existence theorem for these concepts is not so trivial to prove. Nevertheless, it is not difficult and is the only point in the development at which a priority construction is needed.

3.6. **Existence Theorem.** If $\mathcal{A}$ is weakly $(C, \preceq, d)$-priorcomeager under $d$ then there is a $(C, \preceq, d)$-sequence which converges to an element of $\mathcal{A}$.

The direct proof of this theorem proceeds by defining a $(C, \preceq, d)$-sequence which converges and is $(\mathcal{Q}_e)$-prioric, where $(\mathcal{Q}_e)$ is as provided in 3.2. A $(C, \preceq, d)$-sequence $T_0, T_1, \ldots$ is $(\mathcal{Q}_e)$-prioric if for each $e$ there is a number $K(e)$ such that $(T_k)_{k \geq K(e)}$ is $\mathcal{Q}_e$-prioric.

4. **Applications to the $A_1$ theory.** The principal classifications concerning the $A_1$ theory are of the sets $\mathcal{M}, \mathcal{J}$ and $\mathcal{J}$ where $\mathcal{M} = \{B: B \text{ is minimal}\}$, $\mathcal{J} = \{B: B \text{ is incomparable with all nonzero, incomplete } \Sigma^0_1 \text{ degrees}\}$ and $\mathcal{J} = \{B: B^{(1)} > B \cup 0^{(1)}\}$. Namely, $\mathcal{M}$ is $(1, \leq, 0^{(1)})$-priorcomeager, $\mathcal{J}$ is $(C, \leq, 0^{(1)})$-priorcomeager for any system $C$ and $\mathcal{J}$ is weakly $(1, \leq, 0^{(1)})$-priorcomeager. It follows from the existence theorem that $\mathcal{M}$ and $\mathcal{J}$ both contain $A_1$ elements results first obtained by Sacks ([4] and [5]) and Yates [10] respectively.

By involving closure under finite intersections, we can deduce some further results. Using the theorem that every $(C, \leq, 0^{(1)})$-prioromeager set is $(C, \leq, 0^{(1)})$-priorcomeager, it follows that $\mathcal{M} \cap \mathcal{J}$ is $(1, \leq, 0^{(1)})$-priorcomeager and so contains a $A_1$ element, a result first obtained by Sasso [6] and strengthening Shoenfield [9]. Also $\mathcal{M} \cap \mathcal{J}$ is weakly $(1, \leq, 0^{(1)})$-priorcomeager and so contains a $A_1$ element, a result first obtained by Sasso in collaboration with Cooper and Epstein and announced by Sasso [7].

The classification of $\mathcal{M}$ was first obtained in our unpublished lecture notes [11] and has since appeared in [12] along with the classification of $\mathcal{J}$. A classification of $\mathcal{J}$ was obtained in [11], but this involved an awkward modification of the notion of prioromeager set and so was inferior to the result announced here. Another result obtained in [11] was Shoenfield's theorem [8] that if $0^{(1)} \leq c \leq 0^{(2)}$ and $c$ is $\Sigma^0_1$ in $0^{(1)}$ then there is a $b \leq 0^{(1)}$ such that $b^{(1)} = c$. This was the original application of the priority method to the $A_1$ theory (subsequently superseded by Sacks [5] where $b$ was made $\Sigma^0_1$ and requires a notion of a $(C_1, \leq, 0^{(1)})$-prioromeager set for a sequence of systems $(C_e)$ rather than a single system $C$.

5. **Applications to the $\Sigma^0_1$ theory.** The principal classifications concerning the $\Sigma^0_1$ theory are of the sets $\mathcal{P}$ and $\mathcal{P}(a)$ where $\mathcal{P} = \{B: (B)_0 \preceq (B)_1\}$ and $\mathcal{P}(a) = \{B: 0 < B \& a \preceq B\}$. (Here, $(B)_0(n) = B(2n)$ and $(B)_1(n) = B(2n + 1)$ for all $n$; also $a$
is a nonzero $\Sigma_1^0$ degree.) Namely, $\mathscr{P}$ is $(0, \leq^0, 0)$-priorcomeager and $\mathscr{R}(\alpha)$ is weakly $(0, \leq^0, 0)$-priorcomeager under $0$. Either of these results in combination with the existence theorem provides a $\Sigma_1^0$ set of degree strictly between $0$ and $0^{(1)}$; the first of course provides incomparable $\Sigma_1^0$ degrees. Of these, the first result abstracts the original solution to Post’s problem (Friedberg [1] and Mučníc [3]); the second abstracts a technique first introduced by Sacks [4] (this was also abstracted by Lachlan in §2 of [2]).

Finally, it can be shown that both $\mathscr{P}$ and $\mathscr{R}(\alpha)$ are weakly $(0, \leq^a, 0)$-priorcomeager under $0$, where $a$ is a 1-1 recursive function ranging over a $\Sigma_1^0$ set of degree $\alpha$. This shows that there are no minimal $\Sigma_1^0$ degrees, a result first announced by Mučníc.

**Bibliography**


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Section 2

Algebra