Applications of Fourier Integral Operators

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Fourier integral operators, for the calculus of which I refer to Hörmander [17], have been applied in essentially two ways: as similarity transformations and in the description of the solutions of genuinely nonelliptic (pseudo-) differential equations.

The first application is based on the observation of Egorov [12] that if $P$, resp. $Q$, is a pseudo-differential operator with principal symbol equal to $p$, resp. $q$, and $P \circ A = A \circ Q$ for an elliptic Fourier integral operator $A$ defined by the homogeneous canonical transformation $C$, then $p = q \circ C$. This idea, or rather its local version in conic open subsets of the cotangent bundle of the manifolds on which the operators are defined, is a much more powerful tool for bringing operators locally into standard form than merely by coordinate changes in the base space. It has been used not only to reduce the study of wide classes of operators to simple ones like $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} + i x_1 \frac{\partial}{\partial x_2}$, but also in more subtle problems it has been a helpful trick. A rather complete impression of this sort of application can be obtained by looking at the papers of Egorov [13], [14], Nirenberg and Trèves [26], Hörmander [18], Duistermaat and Hörmander [7], Sato, Kawai and Kashiwara [27], Duistermaat and Sjöstrand [8], Sjöstrand [28], Boutet de Monvel [3], and Weinstein [29].

In this respect the following conjecture of Singer seems interesting. Let $L^m(X)$ denote the space of pseudo-differential operators of order $m$, $L^{-\infty}(X)$ the space of smoothing operators. I restrict here to operators for which the total symbol has an asymptotic expansion in homogeneous terms of integer order. If $A$ is an elliptic Fourier integral operator defined by a canonical transformation $C: T^*Y \setminus 0 \to T^*X \setminus 0$, then $P \mapsto A^{-1}PA$ is an isomorphism of filtered algebras:
If moreover $A$ is invertible, then $P \mapsto A^{-1}PA$ is an isomorphism of filtered algebras: $\bigcup_{m \in \mathbb{Z}} L^m(X) \to \bigcup_{m \in \mathbb{Z}} L^m(Y)$.

Now the conjecture is that conversely each isomorphism of filtered algebras of pseudo-differential operators is equal to conjugation by a Fourier integral operator. In this direction Singer and I have proved:

**Theorem.** Let $X$ be compact, $H^1(T^*X \setminus 0, \mathcal{C}) = 0$. Then:

(a) Any isomorphism of filtered algebras

$$\bigcup_{m \in \mathbb{Z}} L^m(X)/L^{-\infty}(X) \to \bigcup_{m \in \mathbb{Z}} L^m(Y)/L^{-\infty}(Y)$$

is either equal to conjugation by an elliptic Fourier integral operator, or by one preceded by the automorphism of $\bigcup_{m \in \mathbb{Z}} L^m(X)/L^{-\infty}(X)$ sending the symbol $\sum_j p_j(x, \xi)$ into the symbol $\sum_j e^{-\pi i j} p_j(x, -\xi)$. Here $p_j(x, \xi)$ denotes the homogeneous term of degree $j$ in the asymptotic expansion of the total symbol.

(b) Any isomorphism of filtered algebras: $\bigcup_{m \in \mathbb{Z} \cup \{\infty\}} L^m(X) \to \bigcup_{m \in \mathbb{Z} \cup \{\infty\}} L^m(Y)$ is equal to conjugation by an invertible continuous linear operator $A: C^\infty(Y) \to C^\infty(X)$.

(c) If $A$ is an invertible continuous linear operator: $C^\infty(Y) \to C^\infty(X)$ and $A^{-1}PA \in L^m(Y)$ for every $P \in L^m(X)$, all $m \in \mathbb{Z}$, then $A$ is an elliptic Fourier integral operator.

The second use of Fourier integral operators, namely as solution operators, goes back to the historical origin of their calculus, because in their local representations

$$ (Au)(x) = \int e^{i\varphi(x, y, \theta)} a(x, y, \theta) u(y) \, dy \, d\theta; $$

the integrands $e^{i\varphi(x, y, \theta)} a(x, y, \theta)$ are the asymptotic oscillatory solutions known from geometrical optics and the W.K.B. method in quantum mechanics. Continuous superposition of such waves resembles the construction of Huygens [19], and the observation that the major contributions only come from the places where the phase function $\varphi$ is stationary as a function of $\theta$ is a counterpart of his well-known principle.

Lax [21] and Ludwig [22] showed that the solution operators for the Cauchy problem for strictly hyperbolic equations have local representations like (1), leading among others to results about the propagation of singularities. (For the case of characteristics of constant multiplicity, see Chazarain [4].)

Then in [16] Hörmander developed a local theory of Fourier integral operators to give a description for small $|t|$ of the unitary operator $U(t) = e^{-itP} = \text{solution of the hyperbolic equation } (i^{-1} \partial/\partial t + P) U = 0, U(0) = I$. Here $P$ is a positive elliptic pseudo-differential operator of order 1 on a compact manifold $X$. From it he obtained the estimate

$$ \# \{j; \lambda_j \leq \lambda\} = (2\pi)^{-n} \text{vol } (B^*X) \cdot \lambda^n + O(\lambda^{n-1}), \quad \lambda \to \infty, $$

for the spectrum $\lambda_1 \leq \lambda_2 \leq \cdots$ of $P$. Here $B^*X = \{(x, \xi) \in T^*X; p(x, \xi) \leq 1\}$ and $p$ denotes the principal symbol of $P$. The improvement over previous results was
that the error term is the best possible for general operators $P$.

However from the global calculus of [17] it follows that $U$, regarded as an operator from $C^\infty(X)$ to $C^\infty(R \times X)$, is a Fourier integral operator of order $-\frac{1}{4}$ defined by the canonical relation

$$C = \{((t, x), (\tau, \xi)), (y, \eta)); \tau + \rho(x, \xi) = 0, (x, \xi) = \Phi^t(y, \eta)\}.$$  

Here $\Phi^t$ is the time $t$ flow of the Hamilton vector field $H_p (= \sum_j \partial \rho/\partial \xi_j \cdot \partial/x_j - \partial \rho/\partial x_j \cdot \partial/\partial \xi_j$ on local coordinates) defined by $p$.

In particular $U(t)$ is for each $t$ a Fourier integral operator of order 0 defined by the canonical transformation $\Phi^t$, a fact which can also be proved by first showing that $e^{itP}Qe^{-itP} \in L^m(X)$ if $Q \in L^m(X)$ and then applying the theorem above. This shows also that $t \mapsto e^{-itP}$ cannot be a smooth family of Fourier integral operators if the order of $P$ exceeds 1, because then

$$\frac{d}{dt}(e^{itP}Qe^{-itP})|_{t=0} = i[P, Q]$$

is of order $> m$ for most $Q \in L^m(X)$, so the order of $e^{itP}Qe^{-itP}$, if it were pseudodifferential operators, would blow up immediately.

The function

$$t \mapsto \hat{\sigma}(t) = \text{Trace } U(t) = \sum_j e^{-it\lambda_j}$$

is a tempered distribution on $R$ and from the global characterization of $U$ as a Fourier integral operator it follows that it can only have singularities at the periods of periodic $H_p$-solution curves. The singularity at a period $T$ can be tested by multiplying $\hat{\sigma}$ with a smooth cut-off function $\rho$, having its support in a small neighborhood of $T$, and then investigating the asymptotic behaviour of the inverse Fourier transform

$$\sum_j \rho(\lambda - \lambda_j) = (\rho * \sigma)(\lambda) = (2\pi)^{-1} \int e^{it\lambda} \rho(t)\hat{\sigma}(t) \, dt$$

as $\lambda \to \infty$.

The analysis of the singularity at $t = 0$ gives back the asymptotic expansion (2) of Hörmander and implies (see [10]) the Minakshisundaram-Pleijel formula for $\sum_j \exp(-\lambda_j^2z)$ (asymptotics for $z \searrow 0$), as well as the statements about the poles of the $\zeta$-function $\sum \lambda_j^{-s}$ in Seeley [30], well known in the case that $P^2 = \text{Laplace operator}$. See also the article by Singer in these PROCEEDINGS.

Chazarain [5] determined the nature of the asymptotic expansion for the singularities at the periods $T \neq 0$ under the assumption of clean intersection of the graph of the $H_p$-flow with the diagonal. In [10] the top order terms of his expansions are computed in terms of the differential of the return map (Poincaré map) of the $H_p$-flow along the periodic solution curve. There it is also shown that periodicity of the total $H_p$-flow with period $T > 0$ is equivalent to a strong asymptotic clustering of the spectrum around the points $2\pi k/T + \beta, k = 1, 2, \ldots$ ($\beta$ is a fixed real number.) It was this clustering effect which destroyed the possibility of improving the
error term in (2). If not all $H_p$-solutions are periodic (and some pathological cases are excluded), then the spectrum is in fact fairly evenly distributed and (2) can be replaced by ($n > 1$):

$$\#\{j; \lambda_j \leq \lambda\} = (2\pi)^{-n} \cdot \text{vol}(B^*X) \cdot \lambda^n$$

$$- (2\pi)^{-n} \left( \int_{S^*X} \text{sub}_P \right) \lambda^{n-1} + o(\lambda^{n-1}), \quad \lambda \to \infty. \quad (6)$$

Here $S^*X = \{(x, \xi) \in T^*X; \ p(x, \xi) = 1\}$, the integration is over the canonical density in $T^*X$ divided by $dp$, and sub $P$ is the so-called subprincipal symbol of $P$.

In the asymptotic expansions for (5) a power of $i$ comes up due to the fact that the principal symbol of $U$ is a section of a complex line bundle (called the Maslov bundle) over $C$, rather than a scalar-valued function. The results of Colin de Verdière [6] suggested that the integer in the exponent should be equal to the Morse index for periodic geodesies if $P = \Delta^{1/2}$. The proof of this resulted in a new approach to the Morse index in variational calculus; see [11].

If the condition of clean intersection is dropped, then the asymptotics of (5) will be that of more general oscillatory integrals, for which the phase functions have degenerate stationary points. Such integrals have been studied by Airy [1], Ludwig [23], and many others. Using Thom's theory of unfoldings of singularities much progress in the understanding of such integrals has been made recently; see Arnol'd [2], Guillemin and Schaeffer [15] and the survey [9]. Malgrange [24] has given an exposition on the relation with the monodromy of singularities.

In [7] parametrices and solutions having their singularities on a bicharacteristic strip have been constructed using Fourier integral operators, for pseudo-differential operators with real principal symbols having only simple zeros or complex principal symbols satisfying an integrability condition.

Recently Melin and Sjöstrand [25] (see also Kučerenko [20]) developed a theory of Fourier integral operators with complex phase functions. They also showed that the projection operators on the kernel and cokernel of operators $P$ such that $\{p, \bar{p}\} \neq 0$ on $p = 0$, constructed in [8], belong to their class. A still more interesting application of their calculus is perhaps the construction of solutions of wide classes of equations with their wave front set on the intersection with the real cotangent bundle of a so-called positive invariant Lagrange manifold.

I hope that I have convinced you that Fourier integral operators now are a well-established tool in the theory of linear partial differential equations. I am sure that its use will continue to have a stimulating effect on the research in this area, at least in the near future.

References


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