Some Minimax Problems in Optimization Theory*

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1. Many problems arising in engineering, economics and mathematics are of the form: Minimize a function \( \varphi(x) \) subject to \( x \in \Omega \) where \( \varphi(x) \) is one of the following functions:

\[
\begin{align*}
(1) & \quad \varphi(x) = \max_{y \in G} f(x, y), \\
(2) & \quad \varphi(x) = \max_{y \in G(x)} f(x, y), \\
(3) & \quad \varphi(x) = \max_{y \in G(x)} \min_{z \in G(x)} f(x, y, z), \\
(4) & \quad \varphi(x) = \max_{y \in G_i(x)} \min_{z \in G_i(x)} \cdots \max_{y_k \in G_k(x)} \min_{z_k \in G_k(x)} f(x, y_1, \cdots, y_k, z_1, \cdots, z_k)
\end{align*}
\]

and sets \( G(x), G_i(x) \) depend on \( x \), \( G \) is a given set.

Such problems often appear in the engineering design theory. In recent years much attention was paid to the problems described. We mention only some books dealing with minimax theory [1], [5], [7], [9], [13]. It seems possible to claim that at present the minimax theory is formed. The minimax theory deals with the following problems:

1. Investigation of the properties of the functions (1)—(4) including their directional differentiability. For various types of functions conditions for the function to be directionally differentiable are obtained and formulae for the first and higher order directional derivatives are found (see for example [6], [8], [9], [10], [18], [20], [21], [22]).

2. Necessary and sufficient conditions and their geometric interpretation [2], [3], [9].

3. Steepest-descent directions and their applications to constructing numerical

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methods. Numerical methods of the first order (of the gradient type). These problems have been widely discussed and studied for the function (1). For this case the first order methods [13] as well as various second order methods (see for example [25]) have been worked out. Some useful estimations have been obtained [12], [16]. Active research is under way to obtain numerical methods for minimizing the function (2) (see [24]) and the function (3) [17]. But there is too much to be done in this field. The main problem for the immediate future is to develop software for minimax problems and its practical applications.

Sometimes it is possible making use of special properties of the problem to develop an effective method for its solution.

4. Saddle points. The problem of finding saddle points is a special case of minimax problems. For this case it is possible to construct methods where it is not necessary to calculate the value of the function (1) at each step (see surveys [11], [14]).

5. Optimal control problems with a minimax criterion function.

6. Nonlinear approximation problems [15], [19].

Now we discuss Problems 1 and 5 in detail.

2. Let $\varphi(x) = \max_{y \in G(x)} f(x, y)$ where $x \in E_n$, $y \in E_m$. Fix $x_0$ and $g \in E_n$. Let

$$\gamma(y) = \{V \in E_m \mid \exists \alpha_0 > 0 : y + \alpha v \in G(x_0 + \alpha g) \forall \alpha \in [0, \alpha_0]\}.$$ 

The closure of $\gamma(y)$ we denote by $\Gamma'(y) = \Gamma'(x_0, y, g)$. It is known [9], [22] that under some additional conditions the function $\varphi(x)$ is differentiable at the point $x_0$ w.r. to the direction of $g$ and

$$\frac{\partial \varphi(x_0)}{\partial g} = \lim_{\alpha \to 0} \frac{1}{\alpha} [\varphi(x_0 + \alpha g) - \varphi(x_0)] = \sup_{y \in R(x_0)} \sup_{v \in I'(y)} \left[\left(\frac{\partial f}{\partial y}, v + \left(\frac{\partial f}{\partial x}, g\right)\right)\right].$$

Higher order derivatives we define as follows. Suppose that $l \geq 2$ and that it is already known that

$$\varphi(x_0 + \alpha g) = \varphi(x_0) + \sum_{k=1}^{l-1} \frac{\partial^k \varphi(x_0)}{\partial g^k} \frac{\alpha^k}{k!} + o(\alpha^{l-1})$$

where $\partial^k \varphi/\partial g^k$ are derivatives of the function $\varphi$ w.r. to the direction of $g$ at the point $x_0$. Then the limit

$$\frac{\partial^l \varphi(x_0)}{\partial g^l} = \lim_{\alpha \to 0} \frac{l!}{\alpha^l} \left[\varphi(x_0 + \alpha g) - \varphi(x_0) - \sum_{k=1}^{l-1} \frac{\partial^k \varphi(x_0)}{\partial g^k} \frac{\alpha^k}{k!}\right],$$

if it exists is called the $l$th derivative of the function $\varphi(x)$ at the point $x_0$ w.r. to the direction of $g$.

Now let us introduce the set $\Gamma'(y, v_1, \ldots, v_{l-1})$ of feasible directions of the $l$th order. Suppose that sets $\Gamma'(y), \Gamma''(y, v_1), \ldots, \Gamma'^{l-1}(y, v_1, \ldots, v_{l-2})$ have already been defined. Fix $y \in G(x_0)$, $v_1 \in \Gamma'(y), \ldots$, $v_{l-1} \in \Gamma'^{l-1}(y, v_1, \ldots, v_{l-2})$ and define the set

$$\gamma'(y, v_1, \ldots, v_{l-1}) = \{v_l \in E_m \mid \exists \alpha_0 > 0 : y + \sum_{k=1}^{l} \alpha_k v_k \in G(x_0 + \alpha g) \forall \alpha \in [0, \alpha_0]\}.$$ 

The closure of $\gamma'(y, v_1, \ldots, v_{l-1})$ let us denote by $\Gamma'(y, v_1, \ldots, v_{l-1})$ and call it the set of feasible directions of the $l$th order (or course $\gamma'$ and $\Gamma'$ depend on $x_0$ and $g$).
Suppose that the function $f$ is $l$ times continuously differentiable. Then for any \( \sigma \in [1 : l] \) the following expansion is valid:

\[
 f(x_0 + \alpha g, y + \sum_{k=1}^{\sigma} \alpha^k v_k + o(\alpha^\sigma)) = f(x_0, y) + \sum_{k=1}^{\sigma} A_k(x_0, y, g, v_1, \ldots, v_k) \frac{\alpha^k}{k!} + o(\alpha^\sigma)
\]

(5)

where \( A_k \) does not depend on \( \sigma \) and is a function of the derivatives of the function \( f \) of order \( \leq k \). Suppose that for any sequence \( \{y_s\} \), \( y_s \in R(x_0 + \alpha_s g) \), \( \alpha_s \to +0 \), there exists a subsequence \( \{y_{s_i}\} \) which can be written as

\[
y_{s_i} = y + \sum_{k=1}^{i-1} \alpha_i^k v_k + \alpha_i v_i + o(\alpha_i^k)
\]

where \( y \in G(x_0), v_1 \in \Gamma^1(y), \ldots, v_{i-2} \in \Gamma^{i-2}(y, v_1, \ldots, v_{i-2}), v_i \in \Gamma^i(y, v_1, \ldots, v_{i-1}), \alpha_i v_i \to 0 \) as \( i \to \infty \).

**Theorem 1.** Let \( l \geq 2 \). If there exists the first derivative of the function \( \phi(x) \) at the point \( x_0 \) w.r. to the direction of \( g \) and (5) is true, then under the assumptions above there exists the derivative of any order \( \sigma \in [2 : n] \) and

\[
 \frac{\partial^{\sigma} \phi(x_0)}{\partial g^{\sigma}} = \sup_{[y,v_1,\ldots,v_{\sigma-1}]} \sup_{T^{\sigma-1}} \A_{\sigma}(x_0, y, g, v_1, \ldots, v_{\sigma})
\]

where \( A_{\sigma}(x_0, y, g, v_1, \ldots, v_{\sigma}) \) is taken from (5); \( T^{\sigma-1} \) is the set of elements of \( [y, v_1, \ldots, v_{\sigma-1}] \) such that supremum in the formula for the \( (\sigma-1) \)th derivative is achieved at points \( [y, v_1, \ldots, v_{\sigma-1}] \). Note that \( T^{\sigma-1} \) is not empty for \( \sigma \in [2 : n] \).

3. **Minimax problems in optimal control.** Let \( \dot{x}(t) = f(x, u, t), x(0) = x_0 \), where \( x = (x_1, \ldots, x_n), f = (f_1, \ldots, f_n), u = (u_1, \ldots, u_r) \) and the functions \( f_i \) and \( \partial f_i / \partial x \) are continuous in all variables. By \( U \) let us denote the class of piecewise continuous controls \( u(t) \) such that \( u(t) \in W \subset C_r \) for any \( t \in [0, T] \). Let \( I(u, z) = \int_0^T g(x, u, t, z) \, dt \) where \( z \in Z \subset E_p \), the functions \( g \) and \( \partial g / \partial x \) are continuous in all variables. Now let us consider the problem

\[
 \max_{z \in Z} \min_{u \in U} I(u, z) \to \min_{u \in U}
\]

(7)

Under some additional conditions the following result is valid.

**Theorem 2 (see [4]).** For a control \( u^* \in U \) to be an optimal one it is necessary that

\[
 \min_{u \in U} \max_{z \in R(u^*)} \int_0^T \left( [H_s(u, \tau) - H_s(u^*, \tau)] \, d\tau \right) = 0
\]

where \( R(u) = \{ z \in Z | I(u, z) = \max_{u \in Z} I(u, v) \} \),

\[
 \frac{d\psi_s(\tau)}{d\tau} = - \left( \frac{\partial f(x(\tau, u^*), u^*, \tau)}{\partial x} \right)^* \psi_s(\tau) - \frac{\partial g(x(\tau, u^*), u^*, \tau, z)}{\partial x}, \psi_s(T) = 0.
\]

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