Convergence in the Maximum Norm of Spline Approximations to Elliptic Boundary Value Problems

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The subject matter in this article is based upon joint work with A. H. Schatz. A somewhat amplified version of this summary may be found in [3] and detailed proofs will be published elsewhere.

Let \( \Omega \) be a bounded domain in Euclidean \( N \)-space \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \). For \( u \) a real valued function defined on \( \Omega \) we shall consider the uniformly elliptic second order differential operator

\[
Lu = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_i} \right) + cu
\]

where \( a_{ij} \) and \( c \) are assumed smooth. The associated bilinear form is given by

\[
B(v, w) = \sum_{i,j=1}^{N} \int_{\Omega} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} \, dx + \int_{\partial \Omega} c v w \, dx.
\]

For \( f \in \mathcal{L}_2(\Omega) \), a weak solution \( u \) of

(1) \( Lu = f \) in \( \Omega \)

satisfies

(2) \( B(u, \varphi) = \int_{\Omega} f \varphi \, dx \equiv (f, \varphi) \)

for all functions \( \varphi \) which are continuous and piecewise continuously differentiable in \( \Omega \) and which vanish near \( \partial \Omega \). We can associate with (1) various kinds of boundary conditions. Examples of these are

(a) \( u = 0 \) on \( \partial \Omega \), or
(b) \( \partial u / \partial \nu = 0 \) on \( \partial \Omega \), or

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\[(c) \frac{\partial u}{\partial n} + u = 0 \text{ on } \partial \Omega; \]

where

\[
\frac{\partial u}{\partial n} = \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_j} n_i.
\]

Here \(n_i\) is the component in the direction \(x_i\) of the outward normal to \(\partial \Omega\) and \(\partial / \partial n\) is called the conormal derivative.

Let \(S_h\) be a linear space of "finite elements" and \(u_h \in S_h\) an approximate solution to the boundary value problem (1) with (a), (b), or (c) satisfied. For many different finite element methods proposed for these problems the interior equations are as follows:

\[
B(u_h, \varphi) = (f, \varphi)
\]

for all \(\varphi \in S_h\) which vanish near \(\partial \Omega\). In many different specific methods which have (3) in common, estimates for norms of the error \(u - u_h\) in Sobolev spaces on all of \(\Omega\) are well known. (For a summary of such results cf. [2].) Interior estimates in Sobolev norms for \(u - u_h\) only satisfying (3) were given in [6] and in maximum norm in [1].

Here we shall consider, instead of \(u_h\) as an approximation to \(u\), certain "local averages of \(u_h\". These, as will be seen subsequently, are formed by computing \(K_h * u_h\), where \(K_h\) is a fixed function and * denotes convolution. As we shall see, the function \(K_h\) has the following properties:

(i) \(K_h\) has small support.

(ii) \(K_h\) is "independent" of the specific choice of \(S_h\) or the operator \(L\).

(iii) \(K_h * u_h\) is easily computable from \(u_h\).

(iv) \(K_h * u_h\) better approximates \(u\) than does \(u_h\).

We shall need some standard notation. To this end denote by \(C^s(\Omega), s = 0, 1, 2, \ldots\), the space of functions defined on \(\Omega\) with uniformly continuous partial derivatives of order up to and including \(s\) on \(\Omega\). For \(v \in C^s(\Omega)\) we set

\[
|v|_{s, \Omega} = \sup_{x \in \Omega} |D^\alpha v(x)|,
\]

where \(\alpha\) is a multi-index, \(|\alpha| = \sum_{i=1}^{N} \alpha_i\), and \(D^\alpha = \partial^{\alpha_1}/\partial x_1^{\alpha_1} \cdots \partial^{\alpha_N}/\partial x_N^{\alpha_N}\). For \(s\) real we define \(H^s(\Omega)\), the Sobolev space with index \(s\) and, for \(v \in H^s(\Omega)\), \(|v|_{s, \Omega}\) will denote its norm (cf. [5]). For example, for \(s = 0, 1, 2, \ldots\), \(|v|_{s, \Omega}\) is given by

\[
||v||_{s, \Omega} = \left( \sum_{|\alpha| \leq s} \int_{\Omega} |D^\alpha v|^2 \, dx \right)^{1/2}.
\]

For \(s\) a positive noninteger \(H^s(\Omega)\) may be defined by interpolating between successive integers and for \(s < 0\) by duality (cf. [5]).

The one-parameter family of spaces \(\{S_h\}_{0 < h \leq 1}\) which we shall consider, will be assumed to have the following properties.

(i) For each \(h, S_h \subset H^1(\Omega)\) and \(S_h\) is finite-dimensional.

(ii) For \(x \in Q_1 \subset \subset \Omega\) and \(U \in S_h\) there are functions \(\varphi_1, \ldots, \varphi_k\) which are piecewise polynomials with compact support such that

\[
U(x) = \sum_{j=1}^{k} \sum_{\alpha \in \mathbb{Z}^n} a_{\alpha} \varphi_j(h^{-1} x - \alpha).
\]
Here $Q \subset \subset \Omega$ means $\bar{Q} \subset \Omega$, $a_\alpha^j$ are real coefficients and $Z^N$ are the multi-integers. (This property may be described as an interior translation invariance property.)

(iii) For some positive integer $r$ there is a constant $C$ such that, for $v \in H^r(\Omega)$, $1 \leq s \leq r$,

$$\inf_{\varphi \in S_h} (\|v - \varphi\|_{0,h} + h\|v - \varphi\|_{1,0}) \leq Ch^r\|v\|_{s,0}.$$  

(iv) Let $Q_1 \subset \subset \Omega$ and let $w$ be an infinitely differentiable function with support in $Q_1$. There is a constant $C$ such that for, $v \in S_h$,

$$\inf_{\varphi \in S_h, \text{ supp } \varphi \subset Q} \|wv - \varphi\|_{1,0} \leq Ch\|v\|_{1,0}.$$  

It was shown in [1] that subspaces consisting of tensor products of one-dimensional splines on a uniform mesh have all the requisite properties. Also in [6] it was demonstrated that the triangular element subspaces in $R^2$ defined in [4] are examples satisfying the above four conditions provided the triangulation is uniform. We emphasize that the uniformity is a condition which is only required locally. Thus we see that many of the finite element subspaces which are discussed in the literature satisfy the above conditions.

In order to define the function $K_h$ we shall need to introduce the so-called smooth splines. In fact we shall choose $K_h$ to be a particular smooth spline depending on the index $r$ associated with the subspace $S_h$.

For $t$ real define

$$\chi(t) = \begin{cases} 1, & |t| \leq \frac{1}{2}, \\ 0, & |t| > \frac{1}{2}, \end{cases}$$

and for $x \in R^N$ set $\phi(x) = \prod_{j=1}^{N} \chi(x_j)$. For $l$ a positive integer set $\phi^{(l)}(x) = (\phi * \cdots * \phi)(x)$, $(l - 1)$ times. The function $\phi^{(l)}$ is just the $N$-dimensional $B$-spline of Schoenberg [7]. The space of smooth splines of order $l$ on a mesh of width $h$ consists of all functions of the form

$$U(x) = \sum_{\alpha \in Z^N} a_\alpha \phi^{(l)}(h^{-1} x - \alpha),$$

for some coefficients $a_\alpha$.

The proof of the following will be given in a forthcoming paper by the author and A. H. Schatz.

**Proposition.** Let $l$ and $t$ be two given positive integers. The smooth spline

$$K_h(x) = \sum_{\alpha \in Z^N} k_\alpha \phi^{(l)}(h^{-1} x - \alpha)$$

may be chosen so that

(a) $k_\alpha = 0$ when $|\alpha_j| > t - 1$ for some $j$,

(b) for $Q_0 \subset \subset Q_1$ and $v \in C^2(Q_1)$ there is a constant $C$ such that

$$\|v - K_h*v\|_{0,0} \leq Ch^{2t} \|v\|_{2t,0},$$

and

(c) for $v \in H^{2t}(Q_1)$ there is a constant $C$ such that

$$\|v - K_h*v\|_{0,0} \leq Ch^{2t} \|v\|_{2t,0}.$$
This function $K_h$ is the aforementioned function in terms of which our local averages will be defined.

Let us denote by $\tilde{S}_h(\Omega_1)$ the subspace of $S_h$ whose elements consist of functions in $S_h$ with support in $\Omega_1$. Let us suppose now that $u_h \in S_h$ satisfies $B(u - u_h, \varphi) = 0$ for all $\varphi \in \tilde{S}_h(\Omega_1)$. These equations are the same as (3) provided $Lu = f$. We now state our main result. The proof of this will be given in a forthcoming paper by the author and A. H. Schatz.

**Theorem.** Let $\Omega_0 \subset \subset \Omega_1 \subset \subset \Omega$ and $p$ be an arbitrary but fixed real number. Let $l = r - 2$ and $t = r - 1$. Then there is a constant $C$ such that, for $u \in H^{2r-2}(\Omega_1)$,
\[
\| u - K_h \ast u_h \|_{0, \Omega} \leq C \{ h^{2r-2} \| u \|_{2r-2, \Omega} + \| u - u_h \|_{-p, \Omega} \}
\]
and, for $u \in H^{2r-2+\left\lfloor N/2 \right\rfloor + 1}(\Omega_1)$,
\[
\| u - K_h \ast u_h \|_{0, \Omega} \leq C \{ h^{2r-2} \| u \|_{2r-2+\left\lfloor N/2 \right\rfloor + 1, \Omega} + \| u - u_h \|_{-p, \Omega} \}.
\]

Let us consider some examples in order to illustrate the meaning of this result. Let $S^r_h$ consist of the smooth splines of order $r$ (restricted to $\Omega$). Then for $\varphi \in S^r_h$ we see that $K_h \ast \varphi \in S^{2r-2}_h$. It is known that, for some $U_h \in S_h(2r-2)$, $u - U_h = O(h^{2r-2})$ as $h \to 0$ for smooth $u$. The theorem says that in fact the special smooth spline $K_h \ast u_h$ is such that $u - K_h \ast u_h = O(h^{2r-2})$ as $h \to 0$ in the interior of $\Omega$ provided that, for some $p$,
\[
(4) \quad \| u - u_h \|_{-p, \Omega} = O(h^{2r-2}).
\]

Let us consider a case where (4) is known to be true for $p = r - 2$. Let $c > 0$ in the operator $L$. Then the solution of the Neumann problem $Lu = f$ in $\Omega$, $\partial u / \partial \gamma = 0$ on $\partial \Omega$ satisfies $B(u, \varphi) = (f, \varphi)$ for all $\varphi \in H^1(\Omega)$. The solution $u_h \in S^r_h$ of $B(u_h, \varphi) = (f, \varphi)$ for all $\varphi \in S^r_h$ exists and is unique. As may be found in [2] the estimate
\[
\| u - u_h \|_{2-r, \Omega} \leq \| u - u_h \|_{2-r, \Omega} \leq C h^{2r-2} \| u \|_{r, \Omega}.
\]
More particularly if we choose $r = 4$ (cubic splines) we obtain, for $N = 2$,
\[
\| u - K_h \ast u_h \|_{0, \Omega} \leq C h^{6} \{ \| u \|_{8, \Omega} + \| u \|_{4, \Omega} \}.
\]
Hence if $u$ is locally smooth and globally less smooth ($u \in H^8(\Omega_1) \cap H^4(\Omega)$) we see that $K_h \ast u_h$ is a local 6th order approximation to $u$ while $u_h$ itself is in general only a 4th order approximation to $u$.

We emphasize that $S_h$ need not be chosen to be the smooth splines (locally) but may be chosen from a much larger class of approximating subspaces of $H^1(\Omega)$.

**References**


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