Absolute Continuity and Singularity of Probability Measures in Functional Spaces

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1. Introduction. The exceptional role played in the probability theory by two random processes—the Wiener and the Poisson ones—naturally poses the following question: What kind of pattern characterises the processes (all of them or a certain class) whose measure is absolutely continuous with respect to the Wiener, and, respectively, the Poisson measure? In more exact terms, let $(C, B)$ be the measurable space of continuous functions $(X_t)_{t \geq 0}$, $X_0 = 0$, $\mathcal{P}$—the Wiener measure, and $\mathcal{P}'$—some other measure on $(C, B)$ which is absolutely continuous with respect to $\mathcal{P}$ ($\mathcal{P}' \ll \mathcal{P}$). The question is now: What is the structure of the random process $\mathcal{X} = (X_t, \mathcal{P}')_{t \geq 0}$? What is the structure of the Radon–Nikodym derivative $d\mathcal{P}' / d\mathcal{P}$—density of $\mathcal{P}'$ with respect to $\mathcal{P}$? If $(D, B)$ is the measurable space of right-continuous piecewise constant functions $(X_t)_{t \geq 0}$, $X_0 = 0$, with $\Delta X_t = 0$ or $+1$ ($\Delta X_t = X_t - X_{t-}$), $\mathcal{P}$—the Poisson measure, and $\mathcal{P}'$—another measure, such that $\mathcal{P}' \ll \mathcal{P}$, find the structure of the random process $\mathcal{X} = (X_t, \mathcal{P}')_{t \geq 0}$ and the derivative $d\mathcal{P}' / d\mathcal{P}$.

Obviously, similar questions arise also for other (not the Wiener or the Poisson) random processes (of a simple structure).

An answer to these and related questions can be obtained within the framework of the general theory of absolute continuity and singularity of the probability measures (ACS), which has made much progress over the past decade. It makes the subject of this paper. It must be stressed that advances in this field are primarily associated with the development of the general theory of random processes, particularly its chapter concerned with the notion of the martingale and its various extensions. "Local martingales, semimartingales, predictability, stochastic integrals with respect to semimartingales and random measures..." that is just a fragment
of the list of notions that researchers have been much excited and preoccupied with over the past ten years or so.

2. ACS criteria. The general problem of the absolute continuity and singularity of probability measures is stated as follows. Let \((\Omega, F, F_t)_{t \geq 0}\) be a measurable space on which is defined a nondecreasing and right-continuous family of \(\sigma\)-algebras \(F_t\) such that \(F = \bigvee F_t\). Let \(P\) and \(\bar{P}\) be two probability measures on \((\Omega, F)\) and

\[
P_t = P|F_t, \quad \bar{P}_t = \bar{P}|F_t
\]

be their restriction on \(F_t\).

We shall say that a measure \(\bar{P}\) is locally absolutely continuous with respect to the measure \(P\) (\(\bar{P} \ll P\)), if \(\bar{P}_t \ll P_t\) for every \(t \geq 0\). The question is: When does it follow from the local absolute continuity that \(\bar{P} \ll P\) (absolute continuity) or that \(\bar{P} \perp P\) (singularity)?

Let \(Z_t = d\bar{P}_t/dP_t\) be the density of \(\bar{P}_t\) with respect to \(P_t\) and \(Z_\infty = \lim_{t \to \infty} Z_t\). The process \(Z = (Z_t, F_t, P)\) is a nonnegative martingale and, therefore, there exists (\(P\)-a.s.) \(\lim_{t \to \infty} Z_t (= Z_\infty)\). One can show that this limit exists also \(\bar{P}\)-a.s.

The following proposition is well known.

**Proposition.** Let \(\bar{P} \ll \text{loc}\ P\). Then

\[
\bar{P} \ll P \iff EZ_\infty = 1,
\]

\[
\bar{P} \perp P \iff EZ_\infty = 0.
\]

The advantage these criteria have over (1) lies with their enabling ACS problem to be reduced to investigating the asymptotic properties of the sequence \((Z_t)_{t \geq 0}\) at \(t \to \infty\) with respect to the measure \(\bar{P}\). By way of illustration, let us show how the well-known absolute continuity/singularity alternative of Kakutani [5] for sequences of independent random variables follows from (2).
Let $P = \mu_1 \times \mu_2 \times \ldots$, $\tilde{P} = \tilde{\mu}_1 \times \tilde{\mu}_2 \times \ldots$ be two measures (which are direct products of measures) in the space of real sequences $x = (x_1, x_2, \ldots)$ and they are such that $\tilde{\mu}_n \ll \mu_n$, $n = 1$. By Kakutani’s result, then the alternative holds: “either $\tilde{P} \ll P$, or $\tilde{P} \perp P$”. This follows directly from (2), since in our case

$$Z_n = \prod_{i=1}^{n} \frac{d\tilde{\mu}_i}{d\mu_i}(x_i)$$

and the $\tilde{P}$-probability of the event $\{Z_\infty < \infty\}$, by the Kolmogorov’s “zero-one” law, has only two values—0 or 1.

From Theorem 1, it is possible also to derive the well-known Hajek–Feldman’s [6], [7] dichotomy which asserts that if $P$ and $\tilde{P}$ are two Gaussian measures in the space of real sequences, then the alternative holds: “they are either equivalent ($\tilde{P} \sim P$), or singular ($\tilde{P} \perp P$”).

The proof of Theorem 1 is simple. Basically, it is a consequence of the Lebesgue decomposition of $\tilde{P}$ with respect to $P$, which, assuming $\tilde{P} \ll \text{loc} P$, appears as

$$\tilde{P}(A) = \int_A Z_\infty dP + \tilde{P}(A \cap \{Z_\infty = \infty\}), A \in F.$$ (3)

(Concerning the proofs of decomposition (3), see [1]-[4]; in [4], applications to Theorem 1 are also given to the question of the validity of the “absolute continuity-singularity” alternative for Markovian sequences.)

3. “Predictable” ACS criteria. Consider in more detail the “local” density $Z = (Z_t, F_t, P)$. First of all, having complemented, if necessary, the $\sigma$-algebra $F_0$ with sets from $F$ of $Q$-measure $(Q = \frac{1}{2} (P + \tilde{P}))$ zero, let us choose a variant of that density with (P-a.s.) regular trajectories (continuous on the right and with left limit). Denote

$$\tau_n = \inf \{t: Z_t < 1/n\}, \tau = \lim_{n} \tau_n$$

and introduce the process

$$M_t = \int_{0}^{t} Z_{s-} \, dZ_s$$ (4)

where $a^\theta$ is a pseudo-inversion of a (i.e., $a^\theta = a^{-1}$ when $a \neq 0$ and $a^\theta = 0$ when $a = 0$). The process $M = (M_t, F_t, P)$ is the $\tau$-local martingale (i.e., $M^n = (M^n_{\tau_n}, F_t, P)$ are martingales for any $n > 1$) and, also, $\Delta M_t \geq -1$ and

$$Z_t = Z_0 + \int_{0}^{t} Z_{s-} \, dM_s, \quad Z_0 = \frac{d\tilde{P}_0}{dP_0}.$$ (5)

This equation has a solution, which is a unique one, in the class of nonnegative local martingales; according to [8], [9], that solution can be written as

$$Z_t = Z_0 \exp \left\{ M_t - \frac{1}{2} \langle M^C \rangle_t + \sum_{s \leq t} \ln (1 + \Delta M_s) - \Delta M_s \right\}$$ (6)
where \( M^c \) is the continuous part of the \( \tau \)-local martingale \( M \) in its decomposition

\[
M = M^c + M^d
\]

(7)

into the continuous and the pure discontinuous components and \( \langle M^c \rangle \) is a characteristic of \( M^c \) (i.e., a predictable increasing process for which \( (M^c)^2 - \langle M^c \rangle \) is the \( \tau \)-local martingale). By virtue of (4) and (5), there is a one-one correspondence between the trajectories \( Z \) and \( M \) (for all \( t < \tau \)). Also, since \( \tilde{P}(\inf Z_t > 0) = 1 \) (cf. the Lebesgue decomposition (3)), \( \tilde{P}(\tau = \infty) = 1 \) and, hence, if we consider \( Z \) and \( M \) with respect to the measure \( \tilde{P} \), there is one-to-one (\( \tilde{P} \)-a.s.) correspondence between them for every \( t > 0 \).

Let now \( \mu \) be the measure of jumps of the process \( M \):

\[
\mu((0, t], \Gamma) = \sum_{s \leq t} I(\Delta M_s \in \Gamma), \ \Gamma \in B(E), \ E = \mathbb{R} \setminus \{0\}
\]

and \( \nu \) the compensator of that measure, [10], [11]. Then decomposition (7) can be written as

\[
M_t = M^c_t + \int_0^t \int_E x \ d(\mu - \nu)
\]

(8)

where \( \int_0^t \int_E x \ d(\mu - \nu) \) is a stochastic integral with respect to the random "martingale" measure \( \mu - \nu \) [12], [3]. From (8), it follows that the predictable [9] characteristics of \( M \) are \( \langle (M^c), \nu \rangle \), and it is with these notions that the ACS conditions can be formulated in a natural way.

As any \( \tau \)-local martingale, the process \( M \) has the property that the square root of the increasing process

\[
[M, M]_t = \langle M^c \rangle_t + \sum_{s \leq t} (\Delta M_s)^2
\]

is \( \tau \)-locally integrable with respect to the measure \( P \) \( ([M, M]^{1/2} \in A_{\text{loc}(\tau)}^+(P)) \), i.e., there exists a sequence \( (\sigma_n)_{n \geq 1} \) of Markov times, \( \sigma_n \uparrow \tau \) and such that

\[
E[M, M]^{1/2}_{\sigma_n} < \infty, \quad n > 1.
\]

In particular,

\[
\sqrt{\sum_{s \in \mathbb{N}} (\Delta M_s)^2} \in A_{\text{loc}(\tau)}^+(P), \quad \langle M^c \rangle \in A_{\text{loc}(\tau)}^+(P).
\]

The condition "with a root" is inconvenient to handle, but one can show [3, Lemma 1] that in fact

\[
\sqrt{\sum_{s \in \mathbb{N}} (\Delta M_s)^2} \in A_{\text{loc}(\tau)}^+(P) \iff \sum_{s \in \mathbb{N}} (\Delta M_s)^2 = 1 + |\Delta M_s| \in A_{\text{loc}(\tau)}^+(P).
\]

Therefore, by virtue of \( M \) being a \( \tau \)-local martingale, the process

\[
\langle M^c \rangle + \sum_{s \in \mathbb{N}} \frac{(\Delta M_s)^2}{1 + |\Delta M_s|} \in A_{\text{loc}(\tau)}^+(P)
\]
or, which is the same,

\[ \langle M^c \rangle + \frac{x^2}{1 + |x|} \mu \in A_{\text{loc}(r)}^+(P) \]  

(9)

where

\[ \frac{x^2}{1 + |x|} \mu \equiv \int_0^t \int_B \frac{x^2}{1 + |x|} \, d\mu = \sum_{s \leq t} \frac{(\Delta M_s)^2}{1 + |\Delta M_s|}. \]

If

\[ \frac{x^2}{1 + |x|} \mu \in A_{\text{loc}(r)}^+(P), \]

then its compensator \([x^2/(1 + |x|)] \nu\) also belongs to the class \(A_{\text{loc}(r)}^+(P)\).

Hence, the process

\[ B(M) \in A_{\text{loc}(r)}^+(P), \]  

(10)

where

\[ B_t(M) = \langle M^c \rangle + \frac{x^2}{1 + |x|} \nu_t. \]  

(11)

The process \(B(M)\) has a fundamental role in the ACS problem since it is its properties that are decisive for the question as to the absolute continuity or singularity of the probability measures \(P\) and \(\bar{P}\). If one examines the properties of the process \(B(M)\) with respect to the measure \(P\), the utmost one can do is to give the sufficient condition for \(P \ll P\). One such condition is as follows ([3, Theorem 12]; see also [13, 14]):

THEOREM 2. Let \(\bar{P} \ll \text{loc} P\) and \(C\) be a constant. Then

\[ P(B_t(M) \ll C) = 1 \Rightarrow P \ll P. \]  

(12)

It is noteworthy, however, that in passing over to the measure \(\bar{P}\) one can obtain the necessary and sufficient conditions both for the absolute continuity and the singularity of the probability measures ([1]-[3]):

THEOREM 3. Let \(\bar{P} \ll \text{loc} P\). Then

\[ \bar{P}(B_\infty(M) \ll \infty) = 1 \Leftrightarrow \bar{P} \ll P, \]  

(13)

\[ \bar{P}(B_\infty(M) = \infty) = 1 \Leftrightarrow \bar{P} \perp P. \]

Let us well on the main points in the proof of that theorem.

By virtue of Theorem 1, we have only to ascertain that \(\bar{P}\)-a.s. \(\{Z_\infty < \infty\} = \{B_\infty(M) < \infty\}\). Since \(\bar{P}(Z_\infty > 0) = 1\), so, by (6), this is equivalent to the proof that \(\bar{P}\)-a.s.

\[ \{\phi \rightarrow\} = \{B_\infty(M) < \infty\} \]

where

\[ \phi_t = M_t - \frac{1}{2} \langle M^c \rangle_t + \sum_{s \leq t} [\ln (1 + \Delta M_s) - \Delta M_s] \]
and \( \{ \varphi \to \} \) is the set of those elementary events for which \( \lim_{t \to \infty} \varphi_t \) exists and is finite.

If \( M \) is a continuous process, then

\[
\varphi_t = M_t - \frac{1}{2} \langle M \rangle_t = (M_t - \langle M \rangle_t) + \frac{1}{2} \langle M \rangle_t
\]

where the process \( N = M - \langle M \rangle \) is with respect to \( \bar{P} \) a continuous local martingale with \( \langle N \rangle = \langle M \rangle \) (\( \bar{P} \)-a.s.). For such martingales it is known (cf., e.g., [3], [15], and a more general statement below in Theorem 4) that \( \{ N \to \} = \{ \langle N \rangle \to \} \) (\( \bar{P} \)-a.s.). Therefore, \( \{ \varphi \to \} = \{ \langle M \rangle \to \} \) (\( \bar{P} \)-a.s., i.e., \( \{ Z_{\infty} < \infty \} = \{ \mathcal{B}_{\infty}^{(M)} < \infty \} \) (\( \bar{P} \)-a.s.).

The general case is somewhat more involved. For considering it, let us introduce the function

\[
u(x) = \begin{cases} x, & |x| \leq 1, \\ \text{sign } x, & |x| > 1 \\ \end{cases}
\]

and pose

\[
\varphi^\nu_t = M_t - \frac{1}{2} \langle M^\nu \rangle_t + \sum_{s \leq t} \left[ u \left( \ln (1 + \Delta M_s) \right) - \Delta M_s \right] .
\]

It must be noted that, while the increments \( \Delta \varphi_t = \ln (1 + \Delta M_t) \) can take on arbitrarily great (modulo) values, the increments \( \Delta \varphi^\nu_t = u(\ln (1 + \Delta M_t)) \) are such that \( |\Delta \varphi^\nu_t| < 1 \).

By a rather simple analysis one sees that \( \bar{P} \)-a.s.

\[
\{ \varphi^\nu \to \} = \{ \varphi \to \}.
\]

So one must only ascertain that \( \bar{P} \)-a.s.

\[
\{ \varphi^\nu \to \} = \{ \mathcal{B}_{\infty} (M) < \infty \}.
\]

Transforming \( \varphi^\nu \), we find that

\[
\begin{align*}
\varphi^\nu_t &= M^\nu_t + M_t - \frac{1}{2} \langle M^\nu \rangle_t + \sum_{s \leq t} \left[ u \left( \ln (1 + \Delta M_s) \right) - \Delta M_s \right] \\
&= M^\nu_t - \frac{1}{2} \langle M^\nu \rangle_t + x * (\mu - v)_t + \left[ u \left( \ln (1 + x) \right) - x \right] * \mu_t \\
&= M^\nu_t - \frac{1}{2} \langle M^\nu \rangle_t + u \left( \ln (1 + x) \right) * (\mu - v)_t + \left[ u \left( \ln (1 + x) \right) - x \right] * v_t \\
&= M^\nu_t - \frac{1}{2} \langle M^\nu \rangle_t + u \left( \ln Y \right) * (\mu - \bar{v})_t + [Y u \left( \ln Y \right) + Y - 1] * v_t \\
&= [M^\nu_t - \langle M^\nu \rangle_t + u \left( \ln Y \right) * (\mu - \bar{v})_t] + \left[ \frac{1}{2} \langle M^\nu \rangle_t + (Y u \left( \ln Y \right) + Y - 1) * v_t \right] \\
&= M^\nu_t - \langle M^\nu \rangle_t + u \left( \ln Y \right) * (\mu - \bar{v})_t + \left[ \frac{1}{2} \langle M^\nu \rangle_t + (Y u \left( \ln Y \right) + Y - 1) * v_t \right] \\
&= (\nu_t + D_t)
\end{align*}
\]

where we have denoted \( Y = 1 + x \) and used the fact that the compensator \( \bar{v} \) of
the measure $\mu$ with respect to $\tilde{P}$ is connected with the compensator $v$ by the expression ([16], [3])

$$d\tilde{\nu} = Ydv.$$ 

At $Y > 0$, the function $Yu(\ln Y) + Y - 1 > 0$ and, since $\Delta M_t \geq -1$, the measure $v$ can be chosen so that $I(Y < 0) * v_\infty = 0$. Hence, $D_t > 0$, and the process $N$ is (with respect to $\tilde{P}$) a locally square-integrable martingale with

$$\langle N \rangle_t = \langle M^c \rangle_t + Y v(\ln Y) * v_t - \sum_{s \leq t} \left[ \int Y u(\ln Y) v(\{s\}, dx) \right]^2.$$ (15)

Hence, and from (14), it follows that the process $\phi^u = N + D$ is a local submartingale with $|\Delta \phi_t| < 1$.

Let us now make use of the following result ([3], [17]):

**Theorem 4.** Let $X = M + A$, where $M$ is a local martingale, $A$ is a non-decreasing process of a locally integrable variation, $A_0 = 0$ and $|\Delta X_t| < C$, $t > 0$. Then (almost surely)

$$\{ A_\infty + \langle M \rangle_\infty < \infty \} = \{ X \to \}.$$ (16)

From this theorem we find that $\tilde{P}$-a.s.

$$\{ D_\infty + \langle N \rangle_\infty < \infty \} = \{ \phi^u \to \}.$$ (17)

Since

$$\sum_{s \leq t} \int Y u(\ln Y) v(\{s\}, dx) = \int_0^t I(v(\{s\}, E) > 0) dD_s$$

so

$$\{ D_\infty + \langle N \rangle_\infty < \infty \} = \left\{ \frac{3}{2} \langle M^c \rangle_\infty + Y v(\ln Y) * v_\infty - \left[ \int_0^\infty I(v(\{s\}, E) > 0) dD_s \right]^2 + [Y u(\ln Y) + Y - 1] * v_\infty < \infty \right\}$$

$$= \left\{ \langle M^c \rangle_\infty + [Y v(\ln Y) + Y u(\ln Y) + Y - 1] * v_\infty - \left[ \int_0^\infty I(v(\{s\}, E) > 0) dD_s \right]^2 < \infty \right\}.$$ 

But

$$\{ D_\infty < \infty \} \subseteq \left\{ \int_0^\infty I(v(\{s\}, E) > 0) dD_s < \infty \right\}$$

and hence

$$\{ D_\infty + \langle N \rangle_\infty < \infty \} = \{ \langle M^c \rangle_\infty + [Y u(\ln Y) + Y u(\ln Y) + Y - 1] * v_\infty < \infty \}.$$ 

At $Y > 0$,

$$Yu^2(\ln Y) + Y u(\ln Y) + Y - 1 \leq (1 - \sqrt{Y})^2$$

and at $x > -1$,

$$(1 - \sqrt{1 + x})^2 \ll \frac{x^2}{1 + |x|}.$$ 

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1 $f \ll g$, if there exist nonzero constants $c_1$ and $c_2$ such that for all argument values $c_1 g < f < c_2 g$. 
Therefore \((\tilde{P}\text{-a.s.})\)
\[
\{\varphi \to \} = \{\varphi^+ \to \} = \{(M^c)_\infty + \left(1 - \sqrt{1 + x}\right)^3 \ast v_\infty < \infty\} \\
= \left\{(M^c)_\infty + \frac{x^2}{1 + |x|} \ast v_\infty < \infty\right\}
\]
which, together with Theorem, proves Theorem 3.

How is Theorem 3 reformulated in the case of discrete time? Let \((\Omega, F)\) be a measurable space, \((F_n)_{n \geq 0}\) a nondecreasing family of \(\sigma\)-algebras such that \(F = \bigvee_n F_n\), \(P\) and \(\tilde{P}\) two probability measures, and \(P_n\) and \(\tilde{P}_n\) their restrictions on \(F_n\), \(\tilde{P}_n \ll P_n\), \(n \geq 0\), \(Z_n = d\tilde{P}_n/dP_n\), \(M_n = \sum_{k=1}^{n} \Delta Z_k\), \(\alpha_n = 1 + \Delta M_n\), \(\mu_n(\Gamma) = I(\Delta M_n \in \Gamma)\), \(v_n(\Gamma) = P(\Delta M_n \in \Gamma | F_{n-1})\), \(\Gamma \in B(E)\). Then \(\tilde{P}\text{-a.s.}\)
\[
B_\infty(M) = \sum_{n=1}^{\infty} E \left[(1 - \sqrt{1 + x})^3 v_n(dx)\right] = \sum_{n=1}^{\infty} E \left[(1 - \sqrt{1 + \Delta M_n})^3 | F_{n-1}\right] \\
= \sum_{n=1}^{\infty} E \left[(1 - \alpha_n)^3 | F_{n-1}\right] = 2 \sum_{n=1}^{\infty} E \left(1 - \sqrt{\alpha_n} | F_{n-1}\right).
\]
Therefore, if \(\tilde{P}_n \ll P_n\), \(n \geq 0\), then
\[
\tilde{P} \ll P \iff \tilde{P} \left\{\sum_{n=1}^{\infty} E \left(1 - \sqrt{\alpha_n} | F_{n-1}\right) < \infty\right\} = 1, \\
\tilde{P} \perp P \iff \tilde{P} \left\{\sum_{n=1}^{\infty} E \left(1 - \sqrt{\alpha_n} | F_{n-1}\right) = \infty\right\} = 1.
\]

In particular, in the case examined by Kakutani the alternative “either \(\tilde{P} \ll P\), or \(\tilde{P} \perp P\)” holds, and also
\[
\tilde{P} \ll P \iff \sum_{n=1}^{\infty} E \left(1 - \sqrt{\alpha_n}\right) < \infty, \\
\tilde{P} \perp P \iff \sum_{n=1}^{\infty} E \left(1 - \sqrt{\alpha_n}\right) = \infty,
\]
where \(\alpha_n = d\tilde{\mu}_n/d\mu_n\).

We shall now formulate two propositions which follow immediately from Theorem 3 (for more detail, see [3]).

Suppose that the \(\tau\)-local martingale has this structure:
\[
M = \gamma^* \cdot m + [(Y^* - 1) + (1 - a)^\oplus (Y^* - a)] \ast (\mu - v), \tag{18}
\]
where \(\mu\) is an integer random measure (not necessarily a measure of the jumps of \(M\)), \(v\) — its compensator, \(m\) — the \(\tau\)-local continuous martingale, \(\gamma^*\) — the predictable process with \(\gamma^* \cdot m < \infty\), \(n \geq 1\), and \(Y^*\) — a \(\tilde{P}\)-predictable process such that \(0 \ll Y^*(t, x) < \infty\), \(Y^*_t = \int_{E} Y^*(t, x) v(dt, dx) < 1\), \(a_t = v((t, E) = Y^*(t, x) = 1\) and
\[
(1 - \sqrt{Y^*})^3 \ast v_m + \sum_{s \leq t_n} I(a_s < 1) \left(1 - \sqrt{\frac{1 - Y^*_s}{1 - a_s}}\right)^2 (1 - a_s) < \infty, \ n \geq 1.
\]
(Hereafter everywhere \((H \cdot X)\), denotes the stochastic integral \(\int_0^t H_s \, dX_s\) with respect to the semimartingale \(X\), [9].)

Now we obtain directly from Theorem 3 this

**CONSEQUENCE 1.** Let \(\tilde{P} \ll \tilde{P} \perp P \Leftrightarrow \tilde{P} \{B^*_\infty < \infty\} = 1\),

\[\tilde{P} \perp P \Leftrightarrow \tilde{P} \{B^*_\infty = \infty\} = 1\]

where

\[B^*_t = (\gamma)^2 \cdot \langle m \rangle_t + (1 - \sqrt{Y^*})^2 \cdot v_t + \sum_{s \leq t} I(a_s < 1) \left(1 - \sqrt{\frac{1 - Y^*_s}{1 - a_s}}\right) (1 - a_s).\]  

**CONSEQUENCE 2.** Let \(\tilde{P} \ll \tilde{P} \perp P, Z_t = d\tilde{P}_t / dP, \mu\) is an integer random measure with the condensator \(v\), \(m\) is the continuous local martingale, and

\(Y^{**} = Z^e \cdot E^P(Z | \phi), \quad \gamma^{**} = Z^e \cdot \frac{d\langle Z^e, m \rangle}{d\langle m \rangle}, \quad B_t^{**} = (\gamma^{**})^2 \cdot \langle m \rangle_t + (1 - \sqrt{Y^{**}})^2 \cdot v_t + \sum_{s \leq t} \left(1 - \sqrt{\frac{1 - a_s}{1 - a_s}}\right) (1 - a_s)\)

with \(a_s = v(\{s\}, E), \quad \tilde{a}_s = \tilde{v}(\{s\}, E)\). Then

\[\tilde{P} \ll P \Rightarrow \tilde{P} \{B^*_\infty < \infty\} = 1.\]

We shall now consider the ACS problem for the particular classes of random processes with continuous time. The approach presented below and based on a combined application of Theorem 2 and Theorem 3 (with its Consequences 1 and 2) has the merit of affording a uniform view of the various specific cases. According to Theorem 3, the decision as to the absolute continuity or singularity implies the ability of finding, in terms of local characteristics, the conditions for local absolute continuity. The examples discussed below demonstrate the manner in which this is done.

4. **Processes with independent increments.** These processes are a natural analogue to sequences of independent random variables, so it will be logical to begin at them.

Under rather general assumptions (cf., e.g., [9], [3]), the process \(X = (X_t, P)\) with independent increments is a semimartingale.

But every semimartingale \(X = (X_t, P)\) admits of the canonical decomposition (P-a.s.)

\[X_t = X_0 + \alpha_t + m_t + \int_0^t \int_{|x| > 1} x \, d\mu + \int_0^t \int_{|x| \leq 1} x \, d(\mu - v),\]

where \(\alpha\) is a predictable process, \(m\) is a continuous local martingale, \(\mu\) is a measurable of the jumps of \(X\), and \(v\) is its compensator. The set \((\alpha, \beta, v)\), where \(\beta = \langle m \rangle\), is referred to as the triplet of predictable characteristics of the semimartingale \(X\).

When the semimartingale \(X\) is a process with independent increments, the triplet \((\alpha, \beta, v)\) is nonrandom. What is essential is that for processes with independent
increments the triplet defines uniquely the distribution of the probabilities \( P \) (in the space of right-continuous and left-hand limited functions), [18].

Let \( \mathcal{F} = (\mathcal{F}_t, \mathcal{F}) \) be another process with independent increments with the triplet \((\bar{\alpha}, \bar{\beta}, \bar{\nu})\). We desire to find the necessary and sufficient conditions on the triplets \((\alpha, \beta, \nu)\) and \((\bar{\alpha}, \bar{\beta}, \bar{\nu})\) for the absolute continuity and singularity of the probability measures \( P \) and \( \bar{P} \).

We shall begin with the necessary conditions. Let \( \bar{P} \ll P \). Then, evidently, \( \bar{P}_0 \ll P_0 \).

In [16], it has been shown that the condition \( \bar{P} \ll P \) entails the fulfilment of the following condition:

\[
\begin{align*}
(\text{i}) & \quad d\bar{\nu} = Y d\nu, \\
(\text{ii}) & \quad \nu(\{t\}, E) = 1 \Rightarrow \bar{\nu}(\{t\}, E) = 1, \\
(\text{iii}) & \quad \langle \bar{m} \rangle = \langle m \rangle, \\
(\text{iv}) & \quad \bar{\alpha}_t - \alpha_t - \int_0^t \int_{|x| \leq 1} x(Y - 1) d\nu = \int_0^t \gamma_s d\langle m \rangle_s
\end{align*}
\]

with

\[
Y = Z^\alpha E^\mu_{\mu}(Z|\bar{\phi}), \quad \gamma = Z^\alpha \frac{d\langle Z^\circ, m \rangle}{d\langle m \rangle}
\]

where \( Z_t = d\bar{P}_t / dP_t \) and \( E^\mu_{\mu}(Z|\bar{P}) \) is the “conditional expectation with respect to the measure \( \mu(dt, dx) P(dw) \) and to the \( \sigma \)-algebra of the \( \bar{\phi} \)-predictable events”, [10].

We now introduce the (deterministic) function \( \bar{B}_t = v(\{t\}, E), \bar{a}_t = \bar{v}(\{t\}, E) \)

\[
B_t = \gamma^2 \cdot \langle m \rangle_t + (1 - \sqrt{Y})^2 \nu_t + \sum_{s \leq t} I(a_s < 1) \left( 1 - \sqrt{\frac{1 - \bar{a}_s}{1 - \bar{a}_s}} \right) (1 - \bar{a}_s)
\]

and formulate the following group of conditions:

\[
\begin{align*}
(\text{iii}) & \quad B_t < \infty, \quad t < \infty, \\
& \quad \text{B}_\infty < \infty, \\
& \quad \text{B}_\infty = \infty.
\end{align*}
\]

The functions \( B^** \) and \( B \), introduced into (20) and (22), respectively, coincide, and, therefore, by Consequence 2 to Theorem 3, \( B_\infty < \infty \).

Thus,

\[
\bar{P} \ll P \Rightarrow \text{I, II, III}_b.
\]

The inverse implication holds as well. Indeed, by virtue of conditions I, II, III_b, the process

\[
M = \gamma \cdot m + [(Y - 1) + (1 - a)^\alpha (\bar{Y} - a)](\mu - v)
\]

is defined, which is a local martingale with \( \Delta M \geq -1 \). Therefore, there exists a nonnegative solution of the equation \( dZ = Z_+ dM \). In the case under consideration,
the function \( B \) is nonrandom and, hence, by virtue of condition III\(_b\) \((B_\infty < \infty)\) and Theorem 2, the family \((Z_t)_{t \geq 0}\) is uniformly integrable with respect to the measure \( P \). Therefore, \( EZ_\infty = 1 \) and the measure \( P' \) with \( dP' = Z_\infty dP \) is probabilistic.

With respect to the measure \( P' \), the process \((X_t)_{t \geq 0}\) also is a process with independent increments. Using the specifics of process (23) and the rules of recalculation of the local characteristics of semimartingales in the case of an absolutely continuous substitution of the measure (cf. Condition II), one can find that the triplet \((\alpha', \beta', \nu')\) of the process \( X' = (X_t, P') \) coincides with the triplet \((\bar{\alpha}, \bar{\beta}, \nu)\). For the processes with independent increments, the triplet defines uniquely the probabilities distribution and, hence, \( P' = \bar{P}, dP' = Z_\infty dP, \) and I, II, III\(_b\) \(\Leftrightarrow\) \( P' \ll P \).

The same method of proof shows that I, II, III\(_b\) \(\Leftrightarrow\) \( P' \ll \text{loc} P \).

Thus, there takes place the following

**THEOREM 5.** If \( X = (X_t, P), \bar{X} = (X_t, \bar{P}) \) are two processes with independent increments, then

1. I, II, III\(_a\) \(\Leftrightarrow\) \( P' \ll \text{loc} P \),
2. if \( \bar{P} \ll \text{loc} P \), then the alternative "either \( \bar{P} \ll P \), or \( \bar{P} \perp P \)" takes place and, in addition,

\[
\text{III}_b \Leftrightarrow \bar{P} \ll P, \\
\text{III}_c \Leftrightarrow \bar{P} \perp P.
\]

Note also that from this proof it follows as well that the density \( Z_t = d\bar{P}_t/dP_t \) is a solution of the equation \( dZ = Z_\infty dM \), where \( M \) is defined by (23).

5. **Semimartingales.** Let \( X = (X_t, P) \) be a semimartingale, i.e., a process admitting of being represented as

\[ X_t = X_0 + A_t + M_t \]

where \( M \) is a local martingale and \( A \) is a process of locally bounded variation. Each semimartingale admits of the canonical representation (21) in which the triplet of local characteristics is, generally speaking, random and does not define uniquely the measure \( P \). It is, indeed, because of this non-uniqueness that one has, apart from such conditions as I, II, III of the preceding section, to introduce, in order to formulate the ACS conditions, one more condition

"(IV) The measure \( \bar{P} \) is \((\sigma_n)\)-unique",

first introduced in [16] and meaning that "stopped" triplets \((\bar{\alpha}_t \wedge \sigma_n, \bar{\beta}_t \wedge \sigma_n, \bar{\nu}(0, t \wedge \sigma_n), dx)\) define uniquely the restrictions \( \bar{P}_{\sigma_n} \) of the measure \( \bar{P} \) on the \( \sigma \)-algebras \( F_{\sigma_n} \), where \( \sigma_n = \inf \{t: B_t > n\} \), and the process \( B \) is defined by (22).

**THEOREM 6.** Let \( X = (X_t, P) \) and \( \bar{X} = (X_t, \bar{P}) \) be two semimartingales and the condition IV be fulfilled. Then

1. I, II, III\(_a\) \(\Leftrightarrow\) \( \bar{P} \ll \text{loc} P \),
2. if \( \bar{P} \ll \text{loc} P \), then

\[
\text{III}_b \Leftrightarrow \bar{P} \ll P, \\
\text{III}_c \Leftrightarrow \bar{P} \perp P.
\]
The proof of the statement I, II, III a $\Rightarrow P \ll \log P$ is the same as in Theorem 5. The proof of the inverse statement involves a difficulty with the application of Theorem 2 (unlike the case of the processes with independent increments); the difficulty is caused by the "randomness" of the function $B$. However, since $\Delta B_t \leq 2$ the random variable $B_{\sigma_n} \ll n+2$ and, consequently, the corresponding family of random variables $(Z_t, t \geq 0)$ is uniformly integrable (Theorem 2). As in Theorem 5, hence it is derived that then $\bar{P}_{\sigma_n} \ll P_{\sigma_n}, n \geq 1$, and, as a consequence to this, $\bar{P} \ll \log P$.

As in the preceding case, the density $Z_t = dP_{\sigma_t}/dP$ is also the solution of the equation $dZ = Z_dM$ with the process $M$ defined by (23).

6. Semimartingales with Gaussian martingale part. Let $X = (X_t, P)$ and $X = (X_t, \bar{P})$ be two semimartingales,

$$X_t = X_0 + M_t, \quad X_t = X_0 + \tilde{A} + \tilde{M}_t$$

whose martingale parts, $M$ and $\tilde{M}$, are Gaussian martingales. The question of the absolute continuity and singularity of the measures $P$ and $\bar{P}$ of such (generally, non-Gaussian) processes has been investigated by a great number of authors ([19]–[27]). The method presented above affords the following results.

Introduce the conditions:

(I)

$$\bar{P}_0 \ll P_0,$$

(a) $\tilde{A} = \gamma \langle \tilde{M} \rangle$,

(II)

(b) $\langle \tilde{M}^c \rangle = \langle M^c \rangle$,

(c) $\langle \tilde{M}^d \rangle = q \cdot \langle M^d \rangle, \quad \langle \tilde{M}^d \rangle = q^{-1} \cdot \langle \tilde{M}^d \rangle$

(a) $\bar{P}(B_t < \infty) = 1, \quad t < \infty$,

(III)

(b) $\bar{P}(B_\infty < \infty) = 1$,

(c) $\bar{P}(B_\infty = \infty) = 1$,

where

$$B_t = \gamma^a \cdot \langle \tilde{M} \rangle_t + \sum_{s \leq t} I(\Delta \langle M \rangle_s > 0)(1 - \varphi_s)^2.$$

**Theorem 7.** The following statements hold:

(1) $I, II, III_a \Rightarrow \bar{P} \ll \log P$,

(2) $I, II, III_b \Rightarrow \bar{P} \ll P$

and if, in addition, $\bar{P}_0 \sim P_0$ and

$$\mathbb{E} \exp (-\gamma \cdot \tilde{M}_\infty - \frac{1}{2} \gamma^2 \cdot \langle \tilde{M} \rangle_\infty) = 1,$$

then $\bar{P} \sim P$. 


THEOREM 8. Let the process $\tilde{A}$ be nonanticipative functional of $X (\tilde{A}_t = \tilde{A}_t(X))$. Then

1. $\text{I, II, III}_a \iff \tilde{P} \ll \text{loc} P$,
2. if $\tilde{P} \ll \text{loc} P$, then $\text{III}_b \iff \tilde{P} \ll P$,
3. $\text{III}_c \iff \tilde{P} \perp P$.

Consider a special case of these statements (for their proofs, see [3]).

Let a Gaussian martingale $M$ and a semimartingale $X = A + M$ be defined on the probability space $(\Omega, F, F_1, P)$. Denote their probabilities distributions by $P_M$ and $P_X$. Then

$$A = \gamma \cdot \langle M \rangle, \quad \gamma^2 \cdot \langle M \rangle_{\infty} < \infty \Rightarrow P_X \ll P_M$$

and the conditions

$$A = \gamma \cdot \langle M \rangle, \quad \gamma^2 \cdot \langle M \rangle_{\infty} < \infty, \quad E \exp \left\{ -\gamma \cdot M_{\infty} - \frac{1}{2} \gamma^2 \cdot \langle M \rangle_{\infty} \right\} = 1$$

are necessary and sufficient for the equivalence of the measures $P_M$ and $P_X$. Moreover,

$$dP_M(X)/dP_X = E \left\{ \exp \left[ -\gamma \cdot M_{\infty} - \frac{1}{2} \gamma^2 \cdot \langle M \rangle_{\infty} \right] \right\} F^x$$

where $F^x = \sigma \{ \omega: X_t, t \geq 0 \}$.

As to the case when $A = A(X)$ is a nonanticipative functional of $X$, the density

$$dP_X/dP_M(X) = \exp \left\{ -\gamma \cdot M_{\infty} - \frac{1}{2} \gamma^2 \cdot \langle M \rangle_{\infty} \right\}.$$ (24)

It is noteworthy that for (arbitrary) Gaussian martingale $M$ the nature of the absolute continuity (and singularity) conditions and the expression for the Radon–Nikodym derivative are the same as when $M$ is a Wiener process [27]. Consider this case in a little more detail. If $W$ is a standard Wiener process, then, as follows from the preceding results, every semimartingale $X = A(X) + W$ with process $A(X)$ such that $A_t(X) = \int_0^t \gamma_s ds, \int_0^\infty \gamma_s^2 ds < \infty$ (P-a.s.) has the measure $P_X$ absolutely continuous with respect to the Wiener measure. In a certain sense, the inverse result also holds (cf. [28], [27]); it is established with the aid of the preceding results and, among other things, gives an answer to the question put forward in § 1.

THEOREM 9. Let a continuous random process $X = (X_t, F_t)$ and a Wiener process $W = (W_t, F_t)$ such that $P_X \ll P^W$ be defined on a complete probability space $(\Omega, F, F_t, P)$. Then there exist such a Wiener process $\tilde{W} = (\tilde{W}_t, F^\tilde{W}_t)$ and a nonanticipative functional $\gamma = \gamma(X)$ that P-a.s.

$$X_t = \int_0^t \gamma_s(X) ds + \tilde{W}_t, \quad t \geq 0.$$
7. Multivariate point processes. Let \((\Omega, F)\) be a measurable space with a non-decreasing family of \(\sigma\)-algebras \((F_t)_{t \in \mathbb{R}^+}\), \(F = \bigvee_{t \in \mathbb{R}^+} F_t\), \((E, \mathcal{E})\) being a Luzin space, \(\Lambda\) an auxiliary point, and \(E_\Lambda = E \cup \{\Lambda\}\), \(\mathcal{E}_\Lambda = \mathcal{E} \vee \{\Lambda\}\). According to [10], [11], a multivariate point process is a term applied to a sequence \((T_n, \xi_n)_{n \in \mathbb{N}}\), where \(T_n\) are Markov times with the properties

1. \(T_0 = 0, \ T_1 > 0,\)
2. \(T_{n+1} > T_n\) if \(T_n < \infty\)
3. \(T_{n+1} = T_n\) if \(T_n = \infty\)

and \(\xi_n\) are \(F_{T_n}\)-measurable random elements with values in \((E_\Lambda, \mathcal{E}_\Lambda)\), and \(X_n = \Lambda\) if and only if \(T_n = \infty\).

Let \(T = \lim_n T_n\) be the point of accumulation.

Every multivariate point process can be conveniently defined by means of the integer random measure \(\mu\) on \((0, \infty) \times E:\)

\[
\mu([0, t] \times \Gamma) = \sum_{n \in \mathbb{N}} I(T_n < t, \xi_n \in \Gamma), \quad \Gamma \in \mathcal{E}.
\]

Pose \(F^n_t = \sigma\{\mu([0, s], B) : s < t, B \in \mathcal{E}\}\), \(G_t = F_0 \vee F^n_t\), \(G = \bigvee_{t \in \mathbb{R}^+} G_t\) and assume that \(F_t = G_t, F = G\). Let, further, \(P\) and \(\bar{P}\) be two probability measures on \((\Omega, G)\) and \(v\) and \(\bar{v}\) be compensators of the measure \(\mu\) with respect to \(P\) and \(\bar{P}\), respectively.

Let us formulate the conditions to be used in Theorem 10 below:

(I) \(\bar{P}_0 \ll P_0,\)
   (a) \(d\bar{v} = Yd\nu\) \((\bar{P}\text{-a.s.}),\)
   (b) \(\nu\{\{t\}, E\} = 1 \Rightarrow \bar{v}\{\{t\}, E\} = 1\) \((\bar{P}\text{-a.s.}).\)

Pose

\[
B_t = (1 - \sqrt{T})^a * \nu_t + \sum_{s \leq t} I(a_s < 1) \left(\left[1 - \sqrt{\frac{1 - \bar{a}_s}{1 - a_s}}\right]^2 (1 - a_s)\right)
\]

where \(a_s = \nu\{\{s\}, E\}\), \(\bar{a}_s = \bar{v}\{\{s\}, E\}\) and let

   (a) \(\bar{P}(B_t = \infty, \ t < T) = 0,\)
   (b) \(\bar{P}(B_T < \infty) = 1,\)
   (c) \(\bar{P}(B_T = \infty) = 1.\)

**Theorem 10.** For multivariate point processes \(X = \{(T_n, \xi_n), P\}\) and \(\bar{X} = \{(T_n, \xi_n), \bar{P}\}\), the following statements hold:

1. \(I, II, III_a \leftrightarrow \bar{P} \ll \text{loc } P;\)
2. if \(\bar{P} \ll \text{loc } P\), then
   \(III_b \leftrightarrow \bar{P} \ll P,\)
   \(III_c \leftrightarrow \bar{P} \perp P.\)
The theorem is proved according to the same method as Theorems 5 and 6. We shall just mention that in the case under consideration—the multivariate point processes—the compensator defines uniquely the probabilities distribution, bearing an analogy to the case of the processes with independent increments, where the triplets of local characteristics, too, defined these distributions uniquely.

From the proof it follows also that the density \( Z_t = d\tilde{P}_t/dP_t, t \geq 0 \), is a solution of the following equation

\[
Z_t = d\tilde{P}_0/dP_0 + Z_\cdot[(Y-1)+(1-a)\circ(Y-a)]*(\mu-v),
\]

where \( \tilde{Y}_s = \int_E Y(s, x) v([s], dx), \quad a_s = v([s], E) \).

An important special case of the multivariate point processes is the so-called point or counting processes for which \( X_n = 1 \).

Denote \( X_t = \mu((0, t], \{1\}) = \sum_{s \leq t} I(T_s < t) \), \( A_t = v((0, t], \{1\}) \), \( \tilde{A}_t = \tilde{v}((0, t], \{1\}) \).

Then condition I will be fulfilled in an evident fashion and condition II will appear as

\[
\begin{align*}
(a) & \quad \tilde{A}_t = \int_0^t Y_s dA_s,
(b) & \quad \Delta A_t = 1 \Rightarrow \Delta \tilde{A}_t = 1 \quad (\tilde{P}\text{-a.s.})
\end{align*}
\]

and the function

\[
B_t = \int_0^t \left( 1 - \sqrt{Y_s} \right)^2 dA_s + \sum_{s \leq t} I(0 < \Delta A_s < 1) \left( 1 - \sqrt{1 - \frac{\Delta \tilde{A}_s}{1 - \Delta A_s}} \right) (1 - \Delta A_s).
\]

An elementary example of a point process is the Poisson process \( X=(X_t, P) \) with parameter equal to one \( (A_t = t) \). Now if \( \tilde{X}=(X_t, \tilde{P}) \) is another point process, then, by Theorem,

\[
\tilde{P} \ll P \iff \tilde{A}_t = \int_0^t Y_s ds, \int_0^\infty \left( 1 - \sqrt{Y_s} \right)^2 ds < \infty \quad (\tilde{P}\text{-a.s.).}
\]

From this result, it follows that the question posed in § 1 is answered: If \( \tilde{X}=(X_t, \tilde{P}) \) is a point process whose measure is absolutely continuous with respect to the Poisson measure, then this process inevitably must have the following structure

\[
X_t = \int_0^t Y_s ds + M_t
\]

where \( M \) is a local martingale and the predictable process \( Y \) is such that \( (\tilde{P}\text{-a.s.}) \int_0^\infty (1 - \sqrt{Y_s})^2 ds < \infty \).

8. Concluding remarks. Theorems 1 and 3 give general and “predictable” criteria of absolute continuity and singularity for two probability measures one of which
is locally absolutely continuous with respect to the other. For the processes with independent increments, semimartingales and multivariant point processes, it has been shown how these criteria are restated in terms of the local characteristics of the processes concerned. As regards the other examples of the efficiency of ACS conditions, the reader is referred to [3]. The corresponding results pertinent to the case of discrete time have been exposed in [1], [4].

References

1. Кабанов Ю. М., Липцер Р. Ш. и Шириев А. Н., К вопросу об абсолютной непрерывности и сингулярности вероятностных мер, Матем. сб. 104, 2 (1977), 227—247.
2. «Предсказуемые» критерии абсолютной непрерывности и сингулярности вероятностных мер (случай непрерывного времени), ДАН СССР, 237, 5 (1977), 1016—1019.
6. Гаек, Я. Об одном свойстве нормальных распределений произвольных стохастических процессов, Чехосл. матем. журн. 8 (1958), 610—618.


23. Гихман И. И. и Скороход А. В., *О плотностях вероятностных мер в функциональных пространствах*, УМН 21 (1966), 83—152.


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