Recursive Enumerability

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One of the fundamental contributions of mathematical logic has been the precise definition and study of algorithms and the closely associated study of recursively enumerable sets. A subset $A \subseteq \omega$ is recursive (decidable) if there is an algorithm for computing its characteristic function $c_A$ and recursively enumerable (r.e.) if there is an algorithm for generating its members. Nonrecursive r.e. sets have played a crucial role in undecidability results beginning with Gödel’s incompleteness theorem [2] and more recently in number theory and group theory. Matiyasevič showed undecidability of Hilbert’s tenth problem by proving that every r.e. set $A$ is Diophantine (namely there is a polynomial $p(x, y)$ with integral coefficients such that $x \in A$ iff $(\exists y)[p(x, y) = 0]$), and Boone, Clapham and Fridman each independently proved that every r.e. degree is the degree of the word problem for a finitely presented group (thus generalizing the Boone–Novikov result that the word problem is unsolvable).

For sets $A, B \subseteq \omega$ (the set of nonnegative integers), $A$ is recursive in (Turing reducible to) $B$, written $A \leq_T B$, if there is an algorithm for computing $c_A$ given $c_B$, and $A \equiv_T B$ if $A \leq_T B$ and $B \leq_T A$. The degree of $A$, $\text{dg}(A)$, is the equivalence class $\{B: B \equiv_T A\}$, $\text{dg}(A) \leq \text{dg}(B)$ if $A \leq_T B$, and a degree is r.e. if it contains an r.e. set. The classification of r.e. sets was initiated by Post [10] who posed the problem: does there exist more than one nonrecursive r.e. degree? The existence of infinitely many such degrees implies for example that there are infinitely many genuinely different unsolvable word problems for finitely presented groups.

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Here we describe some recent results on the classification of r.e. sets and their degrees. A detailed survey of current developments and bibliography can be found in [14], and a complete treatment will appear in the monograph [15].

1. The relation of the structure of an r.e. set to its degree. The r.e. sets \( \{W_n\}_{n \in \omega} \) form a distributive lattice \( \mathcal{E} \) under inclusion whose complemented elements are precisely the recursive sets. The r.e. degrees form an upper semi-lattice \( \mathbb{R} \) with least element \( 0 = \text{dg}(\emptyset) \) and greatest element \( 0' = \text{dg}(K) \), where \( K \) is the complete r.e. set \( \{n : n \in W_n\} \). Post's program for solving his problem was to find a structural property of the complement \( \bar{A} \) of an r.e. set \( A \) which guarantees incompleteness, namely \( \emptyset \prec_T A \prec_T K \). More generally the program is to relate the \( \mathcal{E} \)-structure of an r.e. set to its degree. Post believed that a cofinite r.e. set \( A \) with a complement \( \bar{A} \) sufficiently “thin” with respect to containment of r.e. sets would be incomplete. A cofinite r.e. set \( A \) is simple if \( \bar{A} \) contains no infinite r.e. set. Post constructed simple sets, proved their incompleteness for reducibilities weaker than \( \equiv_T \), and introduced sets with still thinner complements (\( h \)-simple and \( hh \)-simple) to handle \( T \)-reducibility.

Let \( \mathcal{E}^* \) denote the quotient lattice of \( \mathcal{E} \) modulo the ideal \( \mathcal{F} \) of finite sets, and \( A^* \) the equivalence class of \( A \) in \( \mathcal{E}^* \). The thinnest possible (infinite) complement is possessed by a maximal set \( M \), namely \( M \in \mathcal{E} \) such that \( M^* \) is a coatom of \( \mathcal{E}^* \). Friedberg constructed maximal sets and Yates showed they could be complete (i.e. \( M \equiv_T K \)). We give a negative answer to Post's program for a much larger class of properties, those invariant under \( \text{Aut} \mathcal{E} \), the group of automorphisms of \( \mathcal{E} \). (A partial positive answer for noninvariant properties is given in [13].)

**Theorem 1.1.** For any nonrecursive r.e. set \( A \) there exists \( \Phi \in \text{Aut} \mathcal{E} \) such that \( \Phi(A) \equiv_T K \).

Meanwhile, the existence of infinitely many nonrecursive r.e. degrees was shown by Friedberg and Muchnik and their classification under the jump operator was carried out by Sacks, Lachlan, Martin and others. For \( A \subseteq \omega \), define the jump of \( A \), \( A' = \{n : n \in W_n^A\} \) where \( W_n^A \) is the \( n \)th set which is r.e. relative to \( A \). The jump operator is well-defined on degrees, where \( \text{dg}(A)' = \text{dg}(A') \). Let \( a^{(n+1)} = (a^{(n)})' \). For each \( n \geq 0 \) define the subclasses of r.e. degrees,

\[
H_n = \{d : d \in R \text{ and } d^{(n)} = 0^{(n+1)}\}, \quad \text{and} \quad L_n = \{d : d \in R \text{ and } d^{(n)} = 0^{(n)}\},
\]

where \( d^{(0)} = 0 \) and \( L_n = R - L_n \). The degrees in \( H_1(L_1) \) are called high (low) since they have the highest (lowest) possible jump. An r.e. set \( A \) is high (low) if \( \text{dg}(A) \in H_1(L_1) \).

In the opposite direction of Post's approach Martin [8] showed that maximal sets \( M \) (and many others with thin complements) more closely resemble complete than incomplete sets since \( M \) dominates every recursive function, and this guarantees that \( M \) has high degree. For \( \mathcal{G} \subseteq \mathcal{E} \), let \( \text{dg} (\mathcal{G}) = \{\text{dg} (W) : W \in \mathcal{G}\} \).
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THEOREM 1.2 (MARTIN). Let $\mathcal{M}$ be the class of maximal sets. Then $\text{dg}(\mathcal{M}) = H_1$.

If the sets of high degree resemble complete sets, those of low degree should resemble recursive sets. For $A \in \mathcal{E}$ define the principal filter $\mathcal{L}(A) = \{W : W \subseteq \mathcal{E} \text{ and } A \subseteq W\}$. If $R$ is a coinfinite recursive set then $\mathcal{L}(A) \cong \mathcal{S}$, because $R$ is recursively isomorphic to $\omega$.

THEOREM 1.3. If a coinfinite r.e. set $A$ is low ($\text{dg}(A) \in L_1$) then $\mathcal{L}(A) \cong \mathcal{S}$.

A class $C$ of r.e. degrees is invariant if $C = \text{dg}(C)$ for some class $C \subseteq \mathcal{S}$ invariant under $\text{Aut}\mathcal{S}$. Martin asked which other degree classes are invariant besides $H_1$ (and the trivial classes $R, L_0$, and $\tilde{L}_0$). In particular, he asked for a classification of $\text{dg}(\mathcal{M}^*)$ where $\mathcal{M}^*$ is the class of atomless sets, coinfinite r.e. sets with no maximal superset.

THEOREM 1.4 (LACHLAN [4], SHOENFIELD [11]). $\text{dg}(\mathcal{M}^*) = L_2$.

It is unknown whether the methods of this theorem can be combined with Theorem 13 to replace $L_1$ by $L_2$ in the latter. Besides $\tilde{L}_0$, $H_1$, $L_2$, it is unknown which classes of the form $H_n, L_n$ are invariant. The case of $\tilde{L}_1$ is particularly interesting. As a strong generalization of Theorems 1.1 and 1.2 we conjecture that any invariant degree class $C$ is closed upward and $H_1 \subseteq C$.

It was conjectured that every nontrivial invariant degree class $C$ is of the form $H_n$ or $\tilde{L}_n$. This is refuted by the class $\mathcal{D}$ of d-simple sets, coinfinite r.e. sets simple with respect to certain differences of r.e. sets (d.r.e. sets). The latter naturally arise in studying the structure and automorphisms of $\mathcal{S}$ because the members of the Boolean algebra $\mathcal{A}$ generated by $\mathcal{S}$ are finite unions of d.r.e. sets.

THEOREM 1.5 (LERMAN-SOARE [7]). Let $D = \text{dg}(\mathcal{D})$. Then $H_1 \subseteq D$ and $D$ splits $L_1$, so $D$ is not of the form $H_n$ or $\tilde{L}_n$ for any $n$.

2. The structure and automorphisms of $\mathcal{S}$. A major goal in studying the structure of $\mathcal{S}$ is to find complete sets of invariants for classifying the orbit of $A \in \mathcal{S}$ under $\text{Aut}\mathcal{S}$. (Since every $\Phi \in \text{Aut}\mathcal{S}^*$ is induced by some $\Psi \in \text{Aut}\mathcal{S}$ one can consider either $\mathcal{S}$ or $\mathcal{S}^*$.) In [12] a new method is introduced for generating automorphisms of $\mathcal{S}$ and it is used to prove

THEOREM 2.1. If $A$ and $B$ are maximal sets then $\Phi(A) = B$ for some $\Phi \in \text{Aut}\mathcal{S}$.

In the proof we as yet know too little of the structure of $\mathcal{S}$ to specify $\Phi(W)$ immediately given $W \in \mathcal{S}$. Rather we attempt to simultaneously enumerate arrays of r.e. sets $\{W_{f(\alpha)}\}_{\alpha \in \omega}$ and $\{W_{\theta(\alpha)}\}_{\alpha \in \omega}$ such that $\Phi(W_n) = *W_{f(\alpha)}$ and $\Phi^{-1}(W_n) = *W_{\theta(\alpha)}$ (where $X = Y$ denotes that the symmetric difference $X \Delta Y$ is finite). For different values of $\alpha$ these requirements generate considerable conflicts which are resolved by a complicated machinery. An immediate corollary of Theorem 2.1 is that for every $k \in \omega$ the group $\text{Aut}\mathcal{S}^*$ is $k$-ply transitive on its coatoms. Hence, if $\mathcal{S}^*(A)$ is finite then the orbit of $A$ is completely determined.
by the isomorphism type of $\mathcal{L}^*(A)$. When $\mathcal{L}^*(A)$ is infinite this is not necessarily true even when $\mathcal{L}^*(A)$ is particularly well-behaved. We say $X \subseteq Y$ is an $r$-maximal major subset (rm subset) of $Y$ if $Y - X$ is infinite, $Y - X$ is not split into infinite pieces by any recursive set, and for any $W \in \mathcal{E}$, if $W \cup Y = \omega$ then $W \cup X = *\omega$.

**Theorem 2.2 (Lerman–Shore–Soare [5]).** There are r.e. sets $A$ and $B$ such that $\mathcal{L}^*(A)$ and $\mathcal{L}^*(B)$ are isomorphic to the countable atomless Boolean algebra, but $A$ possesses an rm subset while $B$ does not. (Hence, $A$ and $B$ are not automorphic or even elementarily equivalent.)

The proof of Theorem 2.2 relies on a new classification (in terms of $A_3$ functions) of which nonrecursive r.e. sets possess rm subsets, a question arising in the extended decision procedure of Theorem 3.1. In view of Theorem 1.3 another candidate for an easily describable orbit is the class of low simple sets. However, Lerman and Soare showed that there are low simple sets which are $d$-simple and those which are not.

Thus, the Post style classification of $A$ in terms of $\overline{A}$ or $\mathcal{L}(A)$ which has predominated for 30 years is seen to be increasingly inadequate for determining the orbit of $A$. Rather one must examine properties relating $\overline{A}$ to $A$ such as $d$-simplicity and rm subsets. These will give necessary conditions and hopefully the automorphism method of Theorem 2.1 (which relies on the fact that maximal sets possess a strong $d$-simplicity property) will prove the conditions sufficient.

3. The elementary theory of $\mathcal{E}$. One of the most important open questions on $\mathcal{E}$ is the decidability of its elementary theory. Lachlan proved that the theories of $\mathcal{E}$ and $\mathcal{E}^*$ are equi-decidable, and gave a decision procedure for the $\exists \forall \exists$-sentences of the theory of $\mathcal{E}^*$. Lerman and Soare extended this by adding new relations. The aim is to add enough additional relations to give a decision procedure first for the $\exists \forall \exists$-sentences and then perhaps for all sentences.

Let $\mathcal{A}$ be the Boolean algebra generated by $\mathcal{E}$. Let $L$ be the first order language which has function symbols $\cup$, $\cap$, $'$, and a constant symbol 0 to be interpreted in a Boolean algebra as join, meet, complement, and least element respectively, and which has a unary predicate symbol $E(x)$ to be interpreted over $\mathcal{A}^*$ as "$x \in \mathcal{E}^*$". An $\exists \forall \exists$-sentence in this language is one of the form $(\forall x)(\exists y) P(x, y)$ with $P$ quantifier free. Lachlan [3] showed there is an algorithm for deciding which $\exists \forall \exists$-sentences of $L$ are true in $\mathcal{A}^*$ when quantifiers range over $\mathcal{E}^*$. The next step in the decision procedure for $\mathcal{E}^*$ is to consider the $\exists \forall \exists$-sentences of $L$. The most reasonable attack seems to be to expand $L$ by adding new predicates. Let $L^+$ be the result of adding to $L$ the predicates Max $(x)$ and Hhs $(x)$ to be interpreted in $\mathcal{E}^*$ as "$x$ is maximal" and "$x$ is hh-simple". Many new statements become $\exists \forall \exists$ in $L^+$ such as "there exists an atomless hh-simple set with an rm subset", or "there exists an atomless $r$-maximal set". Thus, in addition to a more complicated version of Lachlan’s refinement method new structural theorems were required to prove.
THEOREM 3.2 (LERMAN–SOARE [6]). There is an algorithm for deciding which \( \forall \exists \)-sentences of \( L^+ \) are true in \( \mathcal{A}^4 \) when quantifiers range over \( \mathcal{E}^* \).

Further information is given by classifying the elementary theory of intervals \( \mathcal{L}(A, B) = \{ W : W \in \mathcal{E} \text{ and } A \subseteq W \subseteq B \} \). Stob [18] has shown that if \( A \) is a major subset of \( B \) then the \( \forall \exists \)-theory of \( \mathcal{L}(A, B) \) is decidable and indeed independent of \( A \) and \( B \). The next structural theorems to be proved about \( \mathcal{E} \) will be those needed for further steps in the decision procedure and will not be merely random facts.

4. Relative enumerability. For any degree \( b \) let \( R(b) \) denote the set of degrees \( a \gg b \) such that \( a \) is r.e. relative to \( b \). If \( b \) is low then \( R(b) \) and \( R \) have the same maximum element \( 0' \) but any r.e. degree \( b > 0 \) allows us to obtain new relative r.e. degrees in \( R(b) \).

THEOREM 4.1. For every r.e. degree \( b > 0 \) there is a degree \( a \in R(b) - R \), and \( a \) can be found uniformly in \( b \).

The proof uses a tree of nested strategies construction (as in some recent arguments by Lachlan) to combine the strategies for meeting individual requirements. Since the construction relativizes to any degree \( d \) this negatively answers a conjecture of Cooper that for every high degree \( d \) any degree \( a \gg 0' \) and r.e. in \( 0' \) is r.e. in \( d \). This raises the question of what special role (if any) \( 0' \) plays in \( R(d) \) (in the non-trivial case \( d \ll 0' \) and \( d' > 0' \)). Also open is the homogeneity question of which r.e. degrees \( d \) satisfy \( R(d) \) isomorphic (or even elementarily equivalent) to \( R \).

References

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