Isoperimetric Inequalities and Eigenvalues of the Laplacian

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Given a domain $D$ in euclidean space or on a Riemannian manifold, one naturally associates with it such fundamental geometric quantities as its volume, the measure of its boundary, various curvature functions and their extremes or integrals. One can also associate with the domain $D$ the set of eigenfunctions and eigenvalues of the Laplace operator, subject to given boundary conditions. Recent work has revealed many connections between these two sets of quantities. We shall illustrate just a few of those connections, by concentrating in § I on isoperimetric inequalities between the geometric quantities, and in § II on the Dirichlet boundary value problem. We shall indicate some of the most striking ways in which isoperimetric inequalities are related to the distribution of eigenvalues. For a more detailed account of many of the subjects outlined here, we refer to the author's papers [30], [31].

I. Isoperimetric inequalities. There are many ways in which the classical isoperimetric inequality

\[(1) \quad L^2 \geq 4\pi A \]

has been refined and extended in recent years. Most important have been extensions to domains on surfaces and on higher-dimensional manifolds, or more generally to integral currents and varifolds. Before discussing those, let us note some refinements of (1) for curves in the plane. We shall use the following notation: $D$ denotes a domain, $C$ its boundary, $A$ the area of $D$, and $L$ the length of $C$. Further, let $\rho$ be the inradius of $D$, the maximum radius of open disks lying in $D$, and let $R$ be the circumradius of $D$, the radius of the smallest disk including $D$. 
**THEOREM 1.** Let $D$ be a simply-connected bounded plane domain. For $q < r < R$, one has the following three (equivalent) inequalities:

1. \( rL \geq A + \pi r^2 \),
2. \( L^2 - 4\pi A \geq (L - 2\pi r)^2 \),
3. \( L^2 - 4\pi A \geq (A - \pi r^3)^2/r^2 \).

Inequality (3) is due (in the case of convex domains $D$) to Bonnesen. For a detailed account of Theorem 1 and related results, see a forthcoming paper of the author [30].

The point of inequalities (3) and (4) is that they strengthen the basic isoperimetric inequality (1), and further imply (by choosing, in particular, $r = q$), that equality can hold in (1) only when $D$ is a circular disk.

For our purpose, the case $r = q$ of (2), and its consequence

\[ \frac{L}{A} > \frac{1}{q} \]

will be most important.

A more recent refinement of (1) is due to Sachs [34]. Let $I$ denote the moment of inertia of the curve $C$ with respect to its center of gravity. Then Sachs showed that

\[ L^2 - 4\pi A \geq 4\pi^2 |I/L - A/\pi|. \]

We turn next to domains on surfaces. A general observation is that if one considers domains $D$ of fixed area $A$, then the length $L$ of the boundary tends to increase as the Gauss curvature $K$ decreases. Thus one has the following results.

**THEOREM 2.** Let $D$ be a simply-connected domain of area $A$ bounded by a curve of length $L$. Let $K$ denote Gauss curvature, and let $M = \sup_D K$. Then

\[ L^2 \geq 4\pi A - MA^2, \]

with equality if and only if $K = M$ and $D$ is a geodesic disk.

**THEOREM 3.** Under the same hypotheses, one has

\[ L^2 \geq 4\pi A \left[ 1 - \frac{1}{2\pi} \int_D \int K^+ \right] \]

where $f^+(p)$ denotes $\max \{f(p), 0\}$.

For historical comments concerning these theorems see [31]. A more general inequality is the following. (See Ionin [23] and Burago [7].)

**THEOREM 4.** Let $D$ be a domain with Euler characteristic $\chi$. Let $\lambda$ be any real number, and

\[ \omega_\lambda^+ = \int_D (K - \lambda)^+. \]

Then

\[ L^2 \geq 2A (2\pi \chi - \omega_\lambda^+) - \lambda A^2. \]
When $\chi=1$, (9) induces to (7) for $\lambda=\lambda_0^+$ and to (8) for $\lambda=\lambda_0^-$.

Burago and Zalgaller [8] proved the inequality

$$ql \geq A + \left(\pi - \frac{1}{2} \omega_0^+\right)q^2$$

for simply-connected domains, where $q$ is the maximum distance to the boundary from points of $D$. In the plane case, $q$ is just the inradius, and (10) stands in exactly the same relation to (8) as does (2) (for the case $r=\rho$) to (1). In particular, it implies that (5) continues to hold for simply-connected domains on arbitrary surfaces, provided $\int_{\partial D} K^+ d\sigma \leq 2\pi$.

For surfaces that lie in a larger manifold or in euclidean space, one has other inequalities taking into account the mean curvature of the surface rather than its Gauss curvature. Many of them are based on the formula

$$A = -\int_D \int (x-c) \cdot H dA + \frac{1}{2} \int_C (x-c) \cdot \nu ds$$

where $D$ is a domain on an oriented surface in $\mathbb{R}^n$, $A$ is the area of $D$, $C$ its boundary, $\nu$ the exterior normal to $D$, $H$ the mean curvature vector of the surface, and $c$ is an arbitrary point in $\mathbb{R}^n$. If $C$ consists of a single curve, then one can choose $c$ to be the center of gravity of $C$, which we may take to be the origin, and estimate the right-hand term of (11) by

$$\int_0^L x \cdot \nu ds \leq L^2/2\pi,$$

so that (11) yields

$$L^2 \geq 4\pi \left(A + \int_D \int x \cdot H dA\right)$$

where the origin is at the center of gravity of $C$. In particular, for minimal surfaces one has $H=0$, and the classical isoperimetric inequality (1) holds for minimal surfaces in $\mathbb{R}^n$ bounded by a single curve. Chakerian [12] showed that Sachs’ refinement (6) is also true.

For minimal surfaces whose boundary consists of several curves, one conjectures that (1) continues to hold, but that is not known except for doubly-connected surfaces (Osserman and Schiffer [32] for $n=3$, Feinberg [17] for arbitrary $n$). However, one may still obtain useful inequalities from (11) in the general case. For example, if $D$ lies in a ball of radius $R$, then choosing $c$ to be the center of the ball, one obtains from (11),

$$L \geq \frac{2}{R} A - \int_D |H| dA$$

where equality holds for a plane circular disk of radius $R$.
We consider next higher-dimensional domains. For a domain $D$ in $\mathbb{R}^n$, the analog of (1) is
\begin{equation}
S^n \geq n^n \omega_n V^{n-1}
\end{equation}
where $V$ is the volume of $D$, $\omega_n$ the volume of the unit ball in $\mathbb{R}^n$, and $S$ the $(n-1)$-dimensional measure of the boundary of $V$. Equality holds only for a ball. It was conjectured by Wills [38] that inequality (2) (for the case $r=g$) should generalize to
\begin{equation}
rS \geq V + (n-1) \omega_n q^n
\end{equation}
for convex domains $D$ in $\mathbb{R}^n$, where $q$ is the inradius of $D$. Inequality (15) was proved by Diskant [16], and a stronger form of (15) was given by Osserman [30, Theorem 12] from which it follows that for $n>3$, equality can hold in (15) only for the sphere.\footnote{Added in proof (2/24/79). Professor Wills has called my attention to the fact that his conjecture (15) was also proved independently by J. Bokowski (Elemente der Math. 28 (1973), 43—44).}

For domains on Riemannian manifolds of dimension $n>3$, the analog of (7) is known only for the case of constant sectional curvature (Schmidt [35]) and for geodesic balls in manifolds of variable curvature (Aubin [23]). For many purposes it is sufficient to have weaker inequalities, which although not sharp, give useful upper bounds for the volume of a domain in terms of the area of its boundary. Such inequalities for general Riemannian manifolds and submanifolds have been proved by Schoen [36] and Hoffman and Spruck [22].

The two most interesting open questions seem to be:

1. Does the analog of (7) hold for domains on a Riemannian manifold of variable curvature?
2. Does (14) hold for domains on an arbitrary $n$-dimensional minimal variety in $\mathbb{R}^N$ for any $N>n$?

II. Eigenvalues of the Laplacian. Let $D$ be a plane domain with smooth boundary $C$. The eigenvalue problem
\begin{align}
Du + \lambda u &= 0 \quad \text{in } D, \\
|_{\partial D} &= 0
\end{align}

is known to have a complete system of eigenfunctions $u=\varphi^\lambda_n$, with corresponding eigenvalues $\lambda_n$, where
\[0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots; \quad \lambda_n \to \infty.\]

The basic question is: how are properties of the domain $D$ reflected in the set of eigenvalues $\{\lambda_k\}$? Some of the early results are

\textbf{Theorem 4 (Weyl).}
\begin{equation}
\lim_{n \to \infty} \frac{\lambda_n}{n} = \frac{4\pi}{A}
\end{equation}
where $A$ is the area of $D$.\footnote{Added in proof (2/24/79). Professor Wills has called my attention to the fact that his conjecture (15) was also proved independently by J. Bokowski (Elemente der Math. 28 (1973), 43—44).}
Theorem 5 (Faber-Krahn). Among all domains \( D \) of fixed area \( A \), \( \lambda_1 \) is minimum if and only if \( D \) is a circular disk.

Since \( \lambda_1 \) for a disk of radius \( r \) is known to be \( (j/r)^2 \), where \( j \) is the first positive zero of the Bessel function \( J_0 \), one can state the Faber-Krahn result as

\[
\lambda_1 \geq \frac{\pi j^2}{A}.
\]

In 1957, Peetre [33] showed that inequality (19) holds more generally for domains on a simply-connected surface with \( K \leq 0 \). The following year, Nehari [28] considered vibrations of an inhomogeneous membrane of density \( \rho \), which leads to the equation

\[
\Delta u + \lambda \rho u = 0.
\]

Nehari showed that among all domains of fixed total mass: \( \int_D \rho \, dA \), if \( \log \rho \) is subharmonic, then the minimum of \( \lambda_1 \) for the eigenvalue problem (20), (17), is attained for a circular disk of constant density.

In fact, Peetre's and Nehari's results are exactly equivalent. If we think of \( \rho \) not as a density, but as defining a new conformal metric \( ds^2 = \rho(x, y)(dx^2 + dy^2) \) on \( \mathbb{D} \), then the total mass \( \int_D \rho \, dA \) corresponds to the area of \( \mathbb{D} \) in the new metric, while the Gauss curvature is given by \( K = -(\Delta \log \rho)/2\rho \), so that \( \log \rho \) subharmonic is equivalent to \( K \leq 0 \). Finally, the Laplace-Beltrami operator \( \Delta_s \) with respect to the new metric is given by \( \Delta_s = \Delta/ho \). Thus (20) is just \( \Delta_s u + \lambda u = 0 \), so that the value of \( \lambda_1 \) in Nehari's theorem is the same as \( \lambda_1 \) in Peetre's.

Recently, Bandle has expanded on the common link in Nehari and Peetre, and has obtained a whole series of results including the following [3, p. 205].

Theorem 6. Let \( D \) be a simply connected domain of area \( A \), and let \( M = \sup_D K \). In case \( M > 0 \), assume further that \( A < 4\pi/M \). Let \( D_0 \) be the geodesic disk of area \( A \) on a surface of constant curvature \( M \). Then \( \lambda_1(D) > \lambda_1(D_0) \).

This theorem was also later derived by Chavel and Feldman [13]. All the above results, starting with Faber and Krahn, make use of the various isoperimetric inequalities given above in §1. For example, Theorem 6 uses (7), while Peetre [33, p. 16] also derived from (8) the inequality

\[
\lambda_1 \geq \frac{\pi j^2}{A} \left(1 - \frac{1}{2\pi} \int_D K^+\right)
\]

generalizing (19).

One might think from (19) that \( \lambda_1 \) tends to zero as \( A \) tends to infinity. However one has the result:

Theorem 7. Let \( D \) be a simply-connected plane domain with inradius \( \rho \). Then

\[
\frac{1}{4\rho^2} \leq \lambda_1 \leq \frac{j^2}{\rho^2}, \quad j \sim 2.4.
\]

Thus, for simply-connected domains, \( \lambda_1 \) behaves like the square reciprocal of the inradius.
The right-hand side of (22) follows from an elementary comparison argument. The left-hand side is proved in Osserman [29], using the inequality (5). The first proof of an inequality of the form (22) is due to Hayman [20] who uses a different method and gets a much weaker constant on the left.

By using inequality (10) one can show that the left-hand inequality in (22) holds more generally for simply-connected domains \( D \) on a surface, provided \( \int_D K^+ \leq 2\pi \). The same method can also be used to obtain results for domains of higher connectivity.

**Theorem 8.** Let \( D \) be a plane domain of inradius \( q \) and connectivity \( k \geq 2 \). Then

\[(23) \quad \lambda_1 \equiv 1/kq^2.\]

Recently, Michael Taylor [37] showed that there exists some positive constant \( c > 0 \) such that

\[(24) \quad \lambda_1 > c/kq^2\]

for all plane domains \( D \), where \( k \) is the connectivity of \( D \).

A basic link between isoperimetric inequalities and bounds on \( \lambda_1 \) is provided by a result of Cheeger [14]:

**Theorem 9.** Let \( D \) be a domain on a Riemannian manifold. Set

\[(25) \quad h = \inf_{D'} \frac{S'}{V'},\]

where \( D' \) is a relatively compact subdomain of \( D \), \( V' \) the volume of \( D' \), and \( S' \) the surface area of \( \partial D' \). Then

\[(26) \quad \lambda_1(D) \geq h^2/4.\]

Theorem 7 follows from Cheeger's Theorem and inequality (5), and Theorem 8 follows from a suitable extension of (5). Other inequalities from Part I yield further bounds on \( \lambda_1 \). For example, using (15), one can show [30, Theorem 13] that the left-hand side of (22) is valid for convex domains in \( \mathbb{R}^n \) for all \( n \). Similarly, using (13), one finds

**Theorem 10.** Let \( D \) be a domain on a minimal surface in \( \mathbb{R}^n \). If \( D \) lies in a ball of radius \( R \), then

\[(27) \quad \lambda_1(D) \geq 1/R^2.\]

For \( m \)-dimensional minimal submanifolds of \( \mathbb{R}^n \), one has an exact analog of (13) which yields

\[(28) \quad \lambda_1(D) \geq (m/2R)^3.\]

Another consequence of Cheeger's result is a theorem of McKean [27].

**Theorem 11.** Let \( D \) be a domain on an \( n \)-dimensional simply-connected Riemannian manifold whose sectional curvature is bounded above by \(-\alpha^2, \alpha > 0\). Then

\[(29) \quad \lambda_1(D) \geq [(n-1)\alpha/2]^3.\]
This uses the inequality (7) in the case \( n = 2 \), and an isoperimetric inequality of Yau [39, p. 498] when \( n > 2 \).

Incidentally, inequalities (26) and (29) are both optimal in the following sense. On a surface whose Gauss curvature satisfies \( K \gg -\beta^2 \), \( \beta > 0 \), a geodesic disk \( D_r \) of radius \( r \) satisfies according to Cheng [15]

\[
\lambda_1(D_r) \leq [\beta/2]^2 + [2\pi/r]^2.
\]  

(See also Buser [11] and Gage [19] for sharper versions of (30).) Letting \( r \to \infty \) on a complete surface shows that (29) is a sharp bound. Since (29) was deduced from (26), it follows that the constant \( \frac{1}{4} \) in (26) is best possible. (See also Buser [9].)

As a last application of isoperimetric inequalities, we mention that inequality (1) is used to prove the only known case in which the isospectral problem is solved. The problem is: if two plane domains have the same set of eigenvalues \( \{\lambda_k\} \), are they necessarily congruent? (Can one hear the shape of a drum?) The answer so far is unknown except in the case when one of the domains is a circular disk. In that case one can use Weyl's Theorem (Theorem 4 above) to deduce that both domains have the same area, and then Faber–Krahn (Theorem 5 above) to conclude (from \( \lambda_1 \) alone) that the second domain must also be a disk. For further discussions of this subject, see Berger [6], Fisher [18], Kac [25], [26].

References


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