Shape Theory

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Dedicated to Professor Karol Borsuk

1. Introduction. Shape theory is a new area of topology whose aim is the same as that of homotopy theory, i.e. to study the global behavior of spaces. However, its tools are different and are applicable to rather general spaces. This is not the case with the standard tools of homotopy theory, which are designed primarily for studying such locally nice spaces as CW-complexes and ANR's. E.g., all homotopy groups of the Warsaw circle or of the dyadic solenoid vanish, although these spaces are not contractible. Realizing this, K. Borsuk undertook the task of developing shape theory, a modification of homotopy theory, which agrees with homotopy theory on nice spaces, but yields relevant information even when applied to such general spaces as arbitrary metric compacta. Such spaces appear naturally, e.g., as fibres of maps between nice spaces, and therefore cannot be ignored.

Borsuk's ideas on shape proved to have a bearing on several areas of topology, especially geometric topology, and to have also applications outside of topology. Borsuk's work has triggered an avalanche of research resulting in over four hundred papers written since 1968, when his original paper appeared. It seems therefore appropriate to attempt to survey here the development of shape theory over the past ten years.

Clearly, limitation of space prevents any extensive analysis as well as full attribution of results to all the authors who made relevant contributions. We merely list some major areas and illustrate their nature by a few key theorems. In many cases

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these are not the best results known. Not being able to include a relatively complete bibliography, we just name authors. The reader will easily trace their papers using the extensive shape theory bibliography, compiled by J. Segal, University of Washington, Seattle.

2. The shape category. Originally, Borsuk considered compact metric spaces $X, Y$ embedded in the Hilbert cube $Q$. Instead of homotopy classes of continuous maps $f : X \to Y$, he considered homotopy classes of certain sequences of maps $f_n : Q \to Q$ called fundamental sequences. By definition, for every neighborhood $V$ of $Y$, there is an integer $n_V$ such that all $f_n, n > n_V$, map some neighborhood $U$ of $X$ into $V$ and all $f_n|U, n > n_V$, are homotopic in $V$. Compacta in $Q$ and homotopy classes of fundamental sequences form Borsuk’s shape category $\mathcal{S}h$. Compacta $X$ and $Y$ have the same shape, $\text{sh}X = \text{sh}Y$, provided they are isomorphic objects of $\mathcal{S}h$. Similarly, $X$ is shape dominated by $Y$, $\text{sh}X \rightarrow \text{sh}Y$, provided there exist shape morphisms $f : X \to Y$ and $g : Y \to X$ such that $gf = 1$ in $\mathcal{S}h$.

Subsequently, the shape category $\mathcal{S}h$ has been extended to more general spaces by several authors including S. Mardešić and J. Segal, R. H. Fox, S. Mardešić G. Kozlowski, J. Le Van and C. Weber, and to other categories by W. Holsztyński T. Porter, A. Deleanu and P. Hilton. One should also mention that a 1944 paper by D. E. Christie contains already several ideas of shape theory.

In particular, in 1971 Mardešić and Segal described the shape category $\mathcal{S}h$ for compact Hausdorff spaces using inverse systems of compact ANR’s over cofinite directed sets. If $X=(X_\lambda, p_{\lambda\lambda'}, A)$ and $Y=(Y_\mu, q_{\mu\mu'}, M)$ are such systems and $X=\lim X, Y=\lim Y$, then a shape morphism $X \to Y$ is given by a homotopy class of maps of systems $(f_\mu, \varphi) : X \to Y$, i.e. by an increasing function $\varphi : M \to A$ and maps $f_\mu : X_{\varphi(\mu)} \to Y_{\mu}$ satisfying $f_\mu p_{\varphi(\mu)\varphi(\mu')}(\varphi(\mu')) = q_{\mu\mu'} f_\mu$ whenever $\mu < \mu'$. By definition, $(f_\mu, \varphi) \sim (f_\mu', \varphi')$ if each $\mu$ admits a $\lambda \supseteq \varphi(\mu), \varphi'(\mu)$ such that $f_\mu p_{\varphi(\mu)\lambda} \sim f_\mu' p_{\varphi'(\mu)\lambda}$.

A further generalization of this approach to shape for arbitrary topological spaces was given by K. Morita in 1975. With every topological space $X$ one can associate certain inverse systems $X$ in the homotopy category $\mathcal{H}oo\mathcal{W}$ of spaces having the homotopy type of CW-complexes (or equivalently of ANR’s for metric spaces). Such a system is the Čech system based on numerable coverings. The set of shape morphisms $\mathcal{S}h(X, Y)$ can be interpreted as the set of morphisms $X \to Y$ in the pro-category $\mathcal{P}ro-\mathcal{H}oo\mathcal{W}$ of A. Grothendieck, i.e. as $\lim_{\mu} \colim_\lambda [X_\lambda, Y_\mu]$. Several useful results on pro-categories were developed by M. Artin and B. Mazur in their étale homotopy theory (1969).

If the spaces $X, Y$ have the homotopy type of CW-complexes, (or equivalently of ANR’s), then one can take for $X$ and $Y$ systems which consist of single terms $X$ and $Y$ respectively. Consequently, $\mathcal{S}h(X, Y) = [X, Y]$, i.e. on such spaces shape coincides with homotopy. Another interesting case when shape yields nothing new is
given by the category of compact connected Abelian groups. Their shape morphisms are in a one-to-one correspondence with group homomorphisms. This result, due to J. Keesling, has proved very useful in constructing various counter-examples in shape theory.

3. Homotopy and homology pro-groups. In shape theory the role of the homotopy groups \( \pi_n \) is taken by the homotopy pro-groups \( \text{pro-} \pi_n \). If \((X, x)\) is a pointed \( \mathcal{H} \square \) system associated with \((X, x)\), then \( \text{pro-} \pi_n(X, x) \) is the inverse system of groups \( \left( \pi_n(X, x), p_{n+1} \right) \), where \( p_{n+1} \) is a map. The corresponding inverse limit \( \tilde{\pi}_n(X, x) \) is the \( n \)th shape group of \((X, x)\). For nice spaces, e.g. for ANR's, one can replace the homotopy pro-groups by the shape groups. However, in general much information is lost by passing to the limit and one must consider the whole pro-group as a new and important shape invariant.

This point of view is well illustrated by the following Whitehead theorem (K. Morita, [26]): Let \( f: (X, x) \to (Y, y) \) be a shape morphism of pointed connected finite-dimensional topological spaces (covering dimension based on numerable coverings). If \( f \) induces isomorphisms of homotopy pro-groups in all dimensions, then \( f \) is a shape equivalence. The first result of this type was established by M. Moszyńska [27] and improved by Mardešić and by Morita. D. A. Edwards and R. Geoghegan and also J. Dydak have obtained several theorems of this type weakening the condition that both \( X \) and \( Y \) be finite-dimensional. However, this condition cannot be completely omitted. Indeed, D. Handel and J. Segal have shown that a certain continuum constructed by D. S. Kahn has non-trivial shape although all of its homotopy pro-groups vanish. Kahn's example, just as some other counter-examples in shape theory, depend on J. F. Adams' map \( \Sigma^s \) from the \( 2s \)-fold suspension of a certain finite CW-complex \( Y \) to \( Y \). The crucial property of \( \Sigma^s \) is that for all \( s \), the composition

\[
\Sigma^s \circ \Sigma^s \circ \cdots \circ \Sigma^s \circ \Sigma^s \circ \cdots \circ \Sigma^s \circ A \circ \Sigma^s Y \to Y
\]

is an essential map. Alternatively, one can use results of H. Toda.

Some other standard theorems on homotopy groups also carry over to shape theory and homotopy pro-groups. E.g., this is the case with the van Kampen theorem (A. Kadlof, Š. Ungar) and the Blakers-Massey theorem (Š. Ungar).

Homology pro-groups are defined in an analogous way and their limits are the \( Č \)ech homology groups. In distinction from homology groups, homology pro-groups are exact under very general assumptions. Several versions of the Hurewicz theorem in shape theory have been proved (K. Kuperberg, T. Porter, S. Mardešić and Š. Ungar, K. Morita, Š. Ungar, J. Dydak, T. Watanabe).

4. Stability theorems. An important question in shape theory is to decide when is a pointed space \((X, x)\) stable, i.e. has the pointed shape of a pointed CW-complex.
L. Demers (1975) and Edwards and Geoghegan [9] have shown that a pointed connected space \((X, x)\) is stable if and only if it is pointed shape dominated by a pointed CW-complex. Furthermore, Edwards and Geoghegan have obtained the following **algebraic stability criterion**: A pointed connected space \((X, x)\) of finite shape dimension is pointed stable if and only if each of its homotopy pro-groups \(\text{pro-}\pi_n(X, x)\) is stable, i.e. is isomorphic as a pro-group to a group (which has to be the shape group \(\tilde{\pi}_n(X, x)\)).

(Pointed) metric compacta (pointed) shape dominated by (pointed) CW-complexes are called **(pointed) FANR's** and are the shape theoretic analogues of ANR's. Pointed FANR's are stable, i.e. have the shape of ANR's. However, this is not known in the unpointed case. Are there FANR's which are not pointed FANR’s, has proved to be a very delicate question. It is known that a connected FANR \(X\) is a pointed FANR if and only if the first derived limit \(\lim^1\text{pro-}\pi_1(X, x)\) vanishes (this does not depend on the choice of \(x \in X\)). Some group-theoretic results of J. Dydak and P. Minc indicate that FANR’s might prove to be different from pointed FANR’s.

With every pointed FANR \((X, x)\) Edwards and Geoghegan have associated an intrinsically defined **Wall obstruction** \(w(X, x)\), which takes its values in the reduced projective class group \(\tilde{K}^0(\tilde{\pi}_1(X, x))\) of the first shape group. The vanishing of \(w(X, x)\) is a necessary and sufficient condition for \(X\) to have the shape of a finite CW-complex (or equivalently of a compact ANR). All possible obstructions occur among 2-dimensional FANR’s.

**5. Movability.** One of the most interesting new concepts, which originated in shape theory is the notion of (pointed) movability. A system \(X=(X, p_\lambda, A)\) in \(\mathcal{CC}W\) is said to be **movable** provided each \(\lambda \in A\) admits a \(\lambda' \geq \lambda\) such that for any CW-complex \(K\), any homotopy class of maps \(f: K \to X_{\lambda'}\), and any \(\lambda'' \geq \lambda\) there is a homotopy class of maps \(f': K \to X_{\lambda''}\), such that \(p_{\lambda''}f' = p_{\lambda'}f\). \(X\) is **n-movable** if this holds for complexes \(K\) with \(\dim K \leq n\). A space \(X\) is movable (\(n\)-movable) if the associated systems \(X\) are movable (\(n\)-movable). Movability has proved to be especially useful in the case of metric compacta \(X\), where the vanishing of the shape group \(\tilde{\pi}_1(X, x) = 0\) implies the vanishing of the homotopy pro-group \(\text{pro-}\pi_1(X, x) = 0\). For metric continua \((X, x), (Y, y)\), which are pointed movable, the Whitehead theorem assumes this simple form: If \(X\) and \(Y\) are finite-dimensional and \(f: (X, x) \to (Y, y)\) induces an isomorphism of shape groups, then \(f\) is a shape equivalence (J. Keesling, [16]). Let us also mention that pointed connected FANR’s are characterized as pointed movable continua \((X, x)\) having finite shape dimension and countable shape groups \(\tilde{\pi}_n(X, x)\) (J. Dydak, T. Watanabe).

J. Keesling has obtained interesting results concerning integral Čech cohomology groups \(\tilde{H}^n\) of movable compacta \(X\). In particular, he has proved that \(\tilde{H}^n(X)/\text{Tor}\tilde{H}^n(X)\) is a \(\mathfrak{H}_1\)-free Abelian group, i.e. each of its countable subgroups is free Abelian. Since metric \(LC^{n-1}\) continua (of dimension \(<n\)) are always \(n\)-movable (movable)
6. Shape dimension. A space $X$ is said to have shape dimension (also called deformation dimension) $SD X \leq n$ provided $X$ admits an associated system $X$ whose members $X_k$ are CW-complexes of dimension $\leq n$ (Dydak). It is readily seen that $SD X \leq \text{dim } X$, where $\text{dim }$ is the covering dimension based on numerable coverings of $X$. Moreover, $sh X \leq sh Y$ implies $SD X \leq SD Y$. The main contributions concerning shape dimension are due to S. Nowak. For metric continua $X$, which are shape 1-connected, i.e. $\text{pro-} \pi_1 (X, x) = 0$ for all $x \in X$, and for which $\text{dim } X < \infty$ (or more generally $SD X < \infty$), Nowak has characterized $SD X$ as the greatest integer $n$ such that $H^n(X) \neq 0$. He has also given a characterization theorem for the case when $\text{pro-} \pi_1 (X, x) \neq 0$. Instead of integer coefficients one has to use local systems of groups on members of polyhedral expansions $X$ of $X$.

7. Shapes of $Z$-sets in the Hilbert cube. In [4] T. A. Chapman discovered a profound relationship between shape theory and infinite-dimensional topology. He associated with every $Z$-set $X$ in the Hilbert cube $Q$ its complement $Q \setminus X$ and with every shape morphisms of $Z$-sets $f: X \rightarrow Y$ a class of weakly properly homotopic proper maps $Q \setminus X \rightarrow Q \setminus Y$ in such a manner that one obtains an isomorphism of categories. Using this isomorphism one can translate notions and problems from shape theory into notions and problems concerning weak proper homotopy of separable locally compact spaces. This program has been pursued and extended by Z. Čerin.

Chapman has also proved this remarkable complementation theorem: Two $Z$-sets $X, Y \in Q$ have the same shape if and only if their complements are homeomorphic. Chapman's proof has been simplified by L. Siebenmann.

Stimulated by Chapman's work, D. A. Edwards and H. M. Hastings have introduced a strong (Steenrod) shape theory, which for $Z$-sets in $Q$ transforms under complementation into the more familiar proper homotopy theory. Their approach consists in first organizing inverse systems of semisimplicial complexes and maps of such systems in pro-SS into a closed model category in the sense of D. Quillen (in a way different from the one used by A. K. Bousfield and D. M. Kan). Then, they invert the weak equivalences (in the sense of the calculus of fractions) and obtain a homotopy category $Ho (pro-SS)$ of such systems. With every space $X$ they associate its Vietoris system $V(X)$. Morphisms $V(X) \rightarrow V(Y)$ in $Ho (pro-SS)$ are the strong shape morphisms $X \rightarrow Y$. For compact metric spaces a purely geometric description of strong shape is based on the contractible telescope $C \text{Tel } X$ of an inverse sequence $X$ of compact ANR's. A strong shape morphism $f: X \rightarrow Y$ between metric compacta is just a proper homotopy class of proper maps $f: C \text{Tel } X \rightarrow C \text{Tel } Y$ where $X = \lim X$, $Y = \lim Y$. A systematic study of strong shape of metric compacta, avoiding Quillen's theory, was carried out by Dydk and Segal. Further contributions to strong shape
were made by T. Porter, by Y. Kodama and J. Ono, by F. W. Bauer, by Yu. T. Lisica and by A. Calder and H. Hastings. Beside ordinary shape and strong shape one should mention at least one more shape theory. This is proper shape for locally compact metric spaces, developed by B. J. Ball and R. B. Sher in 1973. Proper shape is the shape analogue of proper homotopy. Among other results Ball and Sher have proved that a proper shape equivalence \( X \to Y \) induces a shape equivalence \( FX \to FY \) between the Freudenthal compactifications, provided these are metrizable. This equivalence reduces to a homeomorphism \( EX \to EY \) on the sets of ends.

8. Finite-dimensional complementation theorems. These are theorems having the following form: If \( X \) and \( Y \) are compacta of dimension \( \leq k \) embedded in a nice way in \( R^n \), \( n > 2k + 2, n > 5 \), then \( \text{sh} \, X = \text{sh} \, Y \) if and only if \( R^n \setminus X \) and \( R^n \setminus Y \) are homeomorphic. The first theorem of this type was proved by Chapman. R. Georghegan and R. Summerhill have improved Chapman's result. Their requirement is that \( R^n \setminus X \) and \( R^n \setminus Y \) be 1-ULC. This result was further improved, first by D. Coram, R. Daverman and P. Duvall, and then by J. Hollingsworth and B. Rushing. They have replaced the condition 1-ULC by the small loop condition (SLC). Finally, replacing SLC by the inessential loop condition (ILC), G. Venema has proved the complementation theorem under the weaker assumption that \( \text{Sd} \, X < k \), \( \text{Sd} \, Y < k \).

We recall that \( X \subseteq R^n \) has ILC provided each neighborhood \( U \) of \( X \) admits a smaller neighborhood \( V \) of \( X \) such that every loop in \( V \setminus X \), which is null-homotopic in \( V \), is also null-homotopic in \( U \setminus X \). Some of these theorems were preceded or supplemented by results dealing with special cases when \( Y \) is \( S^k \) or a manifold (Duvall, Siebenmann, Daverman, Hollingsworth and Rushing, V. T. Liem).

A related problem is the problem of embedding a compactum \( X \) in \( R^n \) up to shape, i.e. of finding a compactum \( Y \subseteq R^n \) such that \( \text{sh} \, X = \text{sh} \, Y \). The first results of this type were obtained by I. Ivanšić. An extensive study of this problem was carried out by L. S. Husch and I. Ivanšić. Here is one of their results. Let \( X \) be an \( r \)-shape connected metric continuum with \( \text{Sd} \, X = k \geq 3 \). If \( X \) is pointed \((r+1)\)-movable, then it embeds up to shape in \( R^{2k-r} \). This generalizes a result of J. Stallings dealing with embedding of polyhedra up to homotopy. Duvall and Husch have recently exhibited a \( k \)-dimensional continuum which does not embed up to shape in \( R^{2k}, k = 2^m, m > 1 \).

Recently, A. Kadlof has answered in the negative the following problem of Borsuk. Let \( \text{sh} \, X < \text{sh} \, Y, Y \subseteq R^n \), does \( X \) embed in \( R^n \) up to shape? Husch and Ivanšić give an affirmative answer under rather restrictive additional assumptions.

9. Cell-like maps. A map \( f \colon X \to Y \) between metric compacta is cell-like (or CE-map) provided each fibre \( f^{-1}(y), y \in Y \), has the shape of a point. These maps have proved
to be extremely important in geometric topology.\(^1\) E.g., J. West has shown that every compact ANR \(Y\) is the image of a compact \(Q\)-manifold \(X\) under a CE-map. If \(X\) and \(Y\) are finite-dimensional (or arbitrary ANR’s) then a CE-map \(f: X \rightarrow Y\) is a shape equivalence. However, in general cell-like maps fail to be shape equivalences. The first counter-example was exhibited by J. L. Taylor (using the Adams map). G. Kozlowski has shown that \(Y\) is an ANR provided \(X\) is an ANR and \(f: X \rightarrow Y\) is a hereditary shape equivalence. An example of J. Keesling shows that one cannot replace this condition by the weaker condition that \(f\) be a CE-map.

Siebenmann and Chapman have proved that CE-maps between \(n\)-manifolds, \(n \geq 5\), and \(Q\)-manifolds respectively, can be approximated by homeomorphisms, i.e. are near-homeomorphisms.

10. Approximate fibrations and shape fibrations. Motivated by R. C. Lacher’s work on cell-like maps, D. Coram and P. Duvall have introduced approximate fibrations. These are maps \(p: E \rightarrow P\) between compact ANR’s satisfying an approximate homotopy lifting property (AHLP). Cell-like maps between ANR’s are always approximate fibrations. Many properties of fibrations are also properties of approximate fibrations. In particular, one has an exact homotopy sequence where the homotopy groups \(\pi_n(F, e)\) of the fibre must be replaced by the shape groups \(\tilde{\pi}_n(F, e)\).

Uniform limits of fibrations between compact ANR’s are readily seen to be approximate fibrations. However, the converse is not true. Husch has considered approximate fibrations \(p: M \rightarrow S^1\), where \(M\) is a closed connected \(n\)-manifold, \(n \geq 6\), and he has proved that \(p\) is a uniform limit of fibrations if and only if the Siebenmann obstruction \(F(M)\) in the Whitehead group \(\text{Wh}(\pi_1(M))\) vanishes. Recently, F. Quinn and Chapman have obtained interesting results concerning the problem of approximating approximate fibrations by block bundles. Further contributions to approximate fibrations were made by S. Ferry and R. E. Goad.

Recently, Mardešić and Rushing have generalized approximate fibrations to a class of maps between metric compacta called shape fibrations. Even in this generality one has a homotopy exact sequence. However, all homotopy groups must be replaced by homotopy pro-groups. Cell-like maps between finite-dimensional compacta, and more generally, hereditary shape equivalences between arbitrary metric compacta, are shape fibrations. The Taylor CE-map is not a shape fibreation. However, every cell-like map is a weak shape fibreation. Further contributions to shape fibrations were made by T. McMillan, A. Matsumoto, M. Jani and Z. Čerin.

11. Shape theoretic properties of the Stone–Čech compactification. Recently, A. Calder and J. Siegel have obtained important new information concerning the homotopy classification of maps from the Stone–Čech compactification \(\beta X\) of

\(^1\) Cf. addresses to this Congress by R. D. Edwards and by J. West. In view of their papers there is no need here for a more detailed exposition concerning CE-maps.
a normal finite-dimensional connected space $X$ into a finite polyhedron $P$. In particular, they have shown that $[\beta X, P]=[X, P]$ provided $\pi_1(P)$ is finite. However, if $\pi_1(P)$ is infinite and $X$ fails to be pseudocompact, then there is no bijection between $[\beta X, P]$ and $[X, P]$. Stimulated by this work, Keesling and Sher have obtained a series of shape theoretic results on $\beta X$, which shed new light on the geometric structure of $\beta X$. E.g., Keesling has proved that if $X$ is Lindelöf and $K \subseteq \beta X \setminus X$ is a continuum, then $\text{sh } K=0$ implies that $K$ is a point. More generally, $\text{Sd } K = \text{dim } K$. If $X$ is real compact, if $K \subseteq \beta X \setminus X$ is a continuum and if $f: K \to Y$ is a surjection, which induces an isomorphism of $H^1$, then $f$ must be a homeomorphism.

References


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