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Geometrical Aspects of Gauge Theories

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1. Introduction. Gauge theories are, broadly speaking, physical theories of a geometrical character such as Einstein's theory of general relativity. In a narrower sense however they correspond to the differential geometry of fibre bundles and were first introduced by H. Weyl in 1916. In recent years they have become increasingly strong candidates to provide a fundamental description of elementary particles. The hope is that, when quantized, gauge theories will explain elementary particles in the same way as quantization of Maxwell theory leads to photons.

The basic difficulty in this programme lies in the nonlinearity of the classical field equations. The successes so far achieved in quantum field theory have depended on sophisticated perturbation techniques which involve expanding about the linearized equations. For many purposes however perturbation theory is inadequate and physicists are exploring alternative nonperturbative approaches to the quantization problem.

One hopeful development has been the discovery that gauge theories possess some remarkable classical solutions which, in simple cases, can be explicitly described. One example is the 'tHooft–Polyakov "magnetic monopole" (see [9]) which is a static solution for an $SU(2)$-gauge theory behaving asymptotically like a usual Dirac monopole (for the group $U(1)$) but having no singularities. Another example is the instanton solution of Belavin et al. [8] for an $SU(2)$-gauge theory in Euclidean 4-space (as opposed to Minkowski space).

The physical significance of instantons is different from that of monopoles and is best understood in terms of the Feynman functional integral approach to quantization. This involves expressing physical quantities as functional integrals involving e^{iS} where S is the (Minkowski space) action or Lagrangian. A standard way to attempt to ascribe meaning to such an integral is to continue analytically into Euclidean space in which case the exponential becomes e^{-S} where S is now the (positive) Euclidean action. Instantons correspond to minima of the Euclidean action. Of course this does not solve the problem of giving a mathematical meaning to the functional integral but it is a step in the right direction and is becoming increasingly popular.

The general hope is in any case that a thorough understanding of the classical nonlinear equations of gauge theories may cast light on the difficult problem of quantization.

2. Instantons. Mathematically instantons, for a given compact Lie group G, correspond to fibre bundles with group G over S^4 (the compactified 4-space, having a connection A which minimizes the L^2-norm $\|F\|^2$ of the curvature F. For G simple and non-abelian (e.g. $SU(2)$) the fibre bundle is classified topologically by an integer k and the minimum of $\|F\|^2$ is $8\pi^2|k|$.

In physical terminology A is the potential, F is the field and $\|F\|^2$ the action. Different integers k correspond to different asymptotic conditions for A in R^4, and k is referred to as the instanton number. Since $\|F\|^2$ depends only on the conformal structure of R^4 it is natural to pass to the conformal compactification S^4.

For $k=1$ and $G=SU(2)$ a solution spherically symmetric about the origin in R^4 was discovered by Belavin et al. [**8**]. This was extended by 'tHooft and others (see [**10**]) to give solutions for $k>1$. Their solutions can be regarded as a nonlinear supposition of k single instantons located at different points of R^4, the superposition being achieved by an ingenious but rather mysterious Ansatz.

A parameter count, using infinitesimal deformation theory, showed that, for $k>2$, this Ansatz did not yield the most general k-instanton [**3**], [**13**], for which more sophisticated methods have had to be used. These methods arise naturally from Penrose's twistor theory [**2**], [**12**] and have led to a complete solution of the instanton problem, not just for $SU(2)$, but for all compact classical groups [**4**].

The geometry underlying Penrose's theory goes back to Plücker and Klein and hinges on the fact that 4-dimensional space can be viewed as the parameter space of lines in 3-space. More precisely, for our purposes, there is a fibration $P_3(C)\to S^4$, where $P_3(C)$ is complex projective 3-space and the fibres are complex projective lines (i.e. 2-spheres). Instanton bundles on S^4 correspond by this map to holomorphic bundles over $P_3(C)$, satisfying suitable constraints [**6**]. Such bundles are necessarily algebraic by a basic theorem of Serre and the constraints are also expressible algebraically (over the real numbers). Thus the instanton problem gets reduced to a problem of real algebraic geometry in 3 dimensions.

Using the powerful techniques of modern algebraic geometry, and following in particular the work of G. Horrocks and W. Barth [**7**], one obtains the complete solution of the instanton problem referred to above. The main technique involved is the use of sheaf cohomology groups, some of which have a direct interpretation in 4-space as solutions of standard linear equations (Laplace, Maxwell, Dirac, etc.). The 'tHooft Ansatz, for example, which is based on Laplace's equation, has a natural cohomological counterpart, and can be generalized to the other equations [6].

The final outcome [4] of this algebraic geometry is a very explicit description of all k-instantons. The solutions, i.e. the connections A and curvature F are given by rational functions of the 4-space coordinates, and depend on a suitable matrix of parameters. The 'tHooft solutions for $SU(2)$ correspond to the special case of a diagonal matrix. The general k-instanton does not have the same local or pointwise appearance due to the presence of the off-diagonal terms.

This solution of the general instanton problem provides a tangible mathematical justification of the Penrose approach. The technical reasons for its success lie in the fact that sheaf cohomology is a more flexible tool than its counterparts in 4-space. In the long run this may well mean that the process of quantization should, as Penrose argues, be carried out in the twistor framework.

There are two caveats that should perhaps be made. In the solution of the instanton problem one key step has to be carried out in the S^4-picture. This involves proving that a certain operator is positive and is not so easy to see in the $P_3(C)$-picture. Secondly the Penrose transformation works well for instantons because these are given by the self-dual Yang–Mills equations (Euler equations for the Lagrangian $\|F\|^2$). The full Yang–Mills equations involve also the anti-self-dual case and because the equations are non-linear we cannot, as in Maxwell theory, combine the two together. However Witten, in a very interesting paper [15] (see also [16]), has recently shown how to interpret the full Yang–Mills equations in a twistor framework. This involves studying the product of $P_3(C)$ with its dual and looking at the 5-dimensional "incidence" hypersurface together with some normal derivatives. Interestingly enough this use of normal derivatives corresponds to a supersymmetric approach involving formal anti-commuting variables.

Finally I should mention a very intriguing paper by Manton [11] which shows that, in a certain precise sense, an infinite superposition of instantons produces a magnetic monopole. To see how this might happen recall that a monopole is time-independent and so can also be viewed as an infinite action solution in Euclidean 4-space.

3. Topological aspects. In the Feynman approach to quantization we have to integrate over the function space of all classical fields. In the Euclidean version of gauge theory, extended to S^4, this space would be the space $\mathscr{A}$ of all connections. There is however a large group of symmetries of this space, namely the group $\mathscr{G}$ of gauge transformations[1] (bundle automorphisms), which preserves the Lag-

[1] For technical reasons it is convenient here to use only gauge transformations equal to the identity at ∞ (i.e. at the base point of S^4).

rangian $\|F\|^2$. Integration should therefore be carried out on the quotient space $\mathscr{C}=\mathscr{A}/\mathscr{G}$. Now $\mathscr{A}$ is a linear space but $\mathscr{C}$ is only a manifold and has to be treated with more respect. Thus for integration purposes a Jacobian term arises which, in perturbation theory, gives rise to the well-known Faddeev–Popov "ghost" particles. Nonperturbatively it seems reasonable that global topological features of $\mathscr{C}$ will be relevant.

Homotopically $\mathscr{C}\sim\Omega^3(G)$ the function space of based maps $S^3\to G$, the components $\mathscr{C}_k$ of $\mathscr{C}$ corresponding to maps of degree k. The k-instantons define a finite-dimensional manifold $M_k\subset\mathscr{C}_k$ given by the minima of the action. For $G=SU(2)$ and $k\to\infty$ it is a remarkable fact [5] that all the homology of $\mathscr{C}_k$ lies in M_k. It is not unreasonable to conjecture that a similar result should hold for all G and that one should even have a homotopy equivalence $M_k\sim\mathscr{C}_k$ as $k\to\infty$. This would tell us that all the global complication in $\mathscr{C}_k$ was already present in M_k (for $k\to\infty$), and might indicate that instanton contributions, suitably interpreted, would converge to the required functional integral.

The results of [5], for $G=SU(2)$, use the 'tHooft solutions depending on configurations of k distinct points, and the striking theorem of G. B. Segal [14] that, for $k\to\infty$, $C_k(R^3)\approx\Omega^3_k(S^3)$ where $C_k(R^3)$ is the configuration space of k points of R^3 and $\approx$ denotes homology equivalence. It is important to note that the fundamental group of $C_k(R^3)$, namely the permutation group Σ_k, gets abelianized on passage to $\Omega^3_k(S^3)$. It seems likely that this already happens in M_k as a result of the nondiagonal matrices in the description of the general k-instanton (for $k\geqslant 3$). This indicates that a "particle" interpretation of the space $\mathscr{C}_k$, while valid for homology purposes, is inadequate in other respects.

Homotopically the space $\Omega^3(G)$ simplifies, for $G=SU(n)$, if we take $n\to\infty$. This suggests that the quantum theory might be soluble in some sense for this limit case and one might then take the $1/n$-expansion to derive information about the finite levels. One very significant feature of the limit theory is that all the homotopy of the function space $\mathscr{C}$ is then contained in the family of (massless) Dirac operators coupled to the (iso-spin representation of the) gauge field. More precisely if we assign to $A\in\mathscr{C}$ the corresponding Dirac operator D_A we get a homotopy equivalence between $\mathscr{C}$ and the space of Fredholm operators in Hilbert space. This is related to the index theorem for elliptic operators and the Bott periodicity theorems concerning the homotopy of the unitary groups [1]. Even foı finite n one can derive interesting consequences [5].

In conclusion, and to put matters in proper perspective, I would like to emphasize that my discussion of geometrical aspects of gauge theories does not imply that geometry or topology alone will unlock the secrets of physics. Many other insights coming from analysis, statistical mechanics and of course experimental physics are needed. The geometrical point of view is however a comparatively new one in the context of quantum field theory and I hope it can provide some useful ideas. It is at least encouraging that the mathematical study of classical gauge theories, geometric, topological and analytic, has over the past decades developed powerful new approaches and techniques. If gauge theories turn out to provide the right explanation of the basic forces of nature, physicists may find that the work of their mathematical colleagues has not been entirely irrelevant.

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