Totally Categorical Structures

§1. Introduction

If an axiomatic theory $T$ has a unique model (up to isomorphism) of a given cardinality $\kappa$, it is said to be $\kappa$-categorical, and if this is the case for all infinite $\kappa$, then the theory is said to be totally categorical. Thus the theory of dense linear orderings having no first or last element is $\aleph_0$-categorical, the theory of algebraically closed fields of specified characteristic is $\aleph_1$-categorical, and a slightly non-trivial example of a totally categorical theory is provided by the theory $A(p^2)$ of abelian groups of exponent $p^2$ in which every element of order $p$ is divisible by $p$.

These three examples are known to be typical in many respects, though the full story is by no means known. As a simple example, it is well known that for any countable model $Q$ of an $\aleph_0$-categorical theory, the automorphism group $G$ of $Q$ has only finitely many orbits on $Q^n$ (for any $n$). This in fact characterizes $\aleph_0$-categorical theories; but the analysis of $\kappa$-categorical theories for $\kappa$ uncountable is more elaborate (as the natural example suggests).

Good results on totally categorical theories in general have only become available quite recently, and the picture is still quite incomplete. The main results are:

THEOREM I. A totally categorical structure $\Lambda$ has the finite submodel property: any first order property of $\Lambda$ is also a property of some finite substructure of $\Lambda$.

THEOREM II. Any totally categorical structure $\Lambda$ can be coordinatized by a collection of geometries (affine, projective, or degenerate).

$^1$ By a convenient abuse of terminology, this means, "a model of a totally categorical theory".
A precise version of this is in § 2. When $A$ is $(Z/p^2Z)^{\infty}$, and $O(p^i)$ ($i = 0, 1, 2$) are the orbits in $A$ under $\text{Aut} A$ (elements of order $p^i$), then $O(p)$ becomes a projective geometry after factoring out an equivalence relation with finite classes, and $O(p^2)$ is fibered over $O(p)$ by affine geometries. The example is typical.

The proofs outlined in § 2 depend indirectly on the finiteness of the set of sporadic finite simple groups. This can be avoided (§ 3).

In the remainder of the paper, I use customary model-theoretical terminology.

§ 2. The structure theory

This outline follows [2], with terminological deviations. All structures will be $\aleph_0$-categorical and $\aleph_0$-stable.

Terminology. An infinite Jordan geometry is a set $S$ equipped with a permutation group $G$ such that for every finite subset $X \subseteq S$: (J) The pointwise stabilizer $G(X)$ of $X$ in $G$ has finitely many orbits, exactly one of which is infinite.

Example. If $S$ is a strongly minimal $\aleph_0$-categorical structure and $G = \text{Aut} S$, then $(S, G)$ is an infinite Jordan geometry.

Jordan geometries are indeed geometries: one can define the closure $\langle X \rangle$ of a finite set $X$ as the union of the finite orbits of $G(X)$, and take the closed sets as subspaces. The geometry is primitive if points are closed, and any geometry can be made primitive by removing $\langle \emptyset \rangle$ and passing to a quotient.

Proposition 1. Infinite primitive Jordan geometries are affine, projective, or degenerate.

The proof is easy, since such a geometry is a limit of finite geometries with a doubly transitive automorphism group, and all large doubly transitive finite groups are known as a consequence of the finiteness of the set of sporadic finite simple groups. (For references see [1] or [2].)

A strongly minimal set associated with a projective or degenerate primitive geometry will be called modular. Two strongly minimal sets contained in a given structure $M$ are orthogonal if the introduction of constants naming the elements of one set has no effect on the geometry of the other (as substructures of $M$). The following result is very powerful, since orthogonal strongly minimal sets are easy to handle.
Proposition 2. There is a 0-definable bijection between the elements of any two nonorthogonal strongly minimal $\aleph_0$ categorical primitive modular structures.

Formulated as a slogan, this becomes: All difficulties are caused by affine geometries.

More terminology (cf. the example following Proposition 3):
1. The structure $M$ is transitive if $\text{Aut } M$ is.
2. If $M, A \subseteq M^*$ with $M$ and $A$ transitive, then $M$ is coordinatized by $A$ (in $M^*$) if:
\[ \forall x \in M \quad \langle x \rangle \cap A \neq \emptyset. \quad (*) \]
3. $A$ is attached to $M$ if there is a structure $M^* 0$-interpretable in $M$ such that $M \cup A \subseteq M^*$. (The underlying set of $M^*$ is a 0-definable subset of $M^n$ for some $n$, taken modulo a 0-definable equivalence relation.)

Proposition 3. Any $\aleph_0$-categorical, $\aleph_0$-stable, transitive $M$ of finite rank $n$ can be coordinatized by a rank one set attached to it.

Example. Let $M$ be the set of all unordered pairs of distinct elements of a set $S$, equipped with the binary relation $R$ defined by: $|p_1 \cap p_2| = 1$. Then $R \subseteq M^2$, and the equivalence relation $E$ on $R$ defined by: $((p_1, p_2), (q_1, q_2)) \in E$ iff $p_1 \cap p_2 = p_3 \cap p_4$ is easily seen to be definable from $R$. $R/E$ is a strongly minimal set attached to and coordinatizing $M$; it can be identified with $S$.

This is a key result. For the proof, pick a rank $(n-1)$ set $S$ whose definition involves parameters $\bar{a}$. We may assume $S$ is normalized in the sense of [3], and varying the parameters get a transitive family $A$ of such sets, with $\bigcup A = M$. The main point is then that rank $A = 1$ (which is perhaps unexpected) and it then follows easily that $A$ coordinatizes $M$. (It is routine to attach $A$ to $M$.)

Using this result, one proves:

Proposition 4. If $M$ is $\aleph_0$-categorical and $\aleph_0$-stable, then $M$ has finite rank.

Theorem II follows.

Theorem I can be derived using Theorem II. If $D \subseteq M$ is a strongly minimal set and $X \subseteq D$ is nonempty, an envelope $E(X)$ is a maximal subset of $M$ independent from $D$ over $X$.

Proposition 5. If $M \models \psi$ and $X$ is large enough, $E(X) \models \psi$. 
Proposition 6. If $X$ is finite then $\text{rank } B(X) < \text{rank } M$. By repeated application of these results, we arrive at Theorem I, generalized to $\aleph_0$-categorical $\aleph_0$-stable structures.

The proof of Proposition 5 consists of an appropriately modified Tarski–Vaught test. If $M \models \varphi(a)$ where the (few) parameters of $\varphi$ all lie in $E(X)$, one can proceed by induction on $\text{rank } (a/E(X))$, to find a similar element in $E(X)$. Theorem II is used here.

§ 3. Historical remarks

It was conjectured in ancient times that a complete $\aleph_1$-categorical theory is never finitely axiomatizable. The conjecture decomposes as follows:

(A) A totally categorical theory has the finite submodel property.

(B) An $\aleph_1$-categorical theory which is not $\aleph_0$-categorical is never finitely axiomatizable.

Peretyat’kin refuted (B) in [10]; subsequently both Morley and Parigot came up with noticeably simpler variants (unpublished). The conjecture (A) was proved by Zil’ber in [12] modulo a serious gap caused by over-simplifying the application of Lemma 5.1. The gap can be filled by a number-theoretic argument. (As far as I know, Zil’ber has not yet published this correction.) Much of the material in [13] was already in [12], in less elegant form (the Parisian viewpoint [6, 7] was taking root in the meantime).

The work in [2] began when the gap in [13] still seemed serious. Lachlan had pointed out the relevance of certain incidence geometries called pseudoplanes to some conjectures about $\aleph_0$-categorical theories [3], and Zil’ber established their relevance to the classification of strongly minimal $\aleph_0$-categorical structures in [16 I]. The model-theoretic form of Proposition 1 was conjectured in [16 II], and proved in [16 II, III] by a direct method using a lot of model theory. This last work dates to summer 1980.

I studied [13] in 1980 during Model Theory Year at the I.A.S. in Jerusalem. In the fall I noticed the truth and relevance of Proposition 1, having just heard of the completion of the classification of finite simple groups. As I was very conveniently surrounded by model theorists at the time, it didn’t take very long for the picture presented here to come into focus. (A graduate student in logic at Wisconsin noticed the truth of Proposition 1 about the same time [9].)

My exposition slights the detailed theory of envelopes [2, § 7], which has been pushed further in [8]. The notion was introduced in [13]. Lachlan came upon a particularly transparent special case in his study of stable

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2 See Note 2.
structures admitting elimination of quantifiers in a finite relational language [4], a special case of “disintegrated” structures (\(\aleph_0\)-categorical, \(\aleph_0\)-stable, with all coordinatizing geometries degenerate). In particular, the fundamental conjecture that \(\aleph_0\)-categorical \(\aleph_0\)-stable structures should be infinite analogs of certain finite structure can be made entirely precise in this context, giving the main conjecture of [4], proved for binary languages in [5].

Continuing this historical outline into the near future, a proof of this conjecture based on finite permutation group theory of the type outlined in [1] is under construction (January 1983). If successful, it should provide further evidence of the usefulness of group-theoretic methods in this area of model theory.

The geometric analysis of sets of rank \(\alpha\) inside a set of rank \(\alpha + 1\), which dominates the proof of Proposition 3, is being pursued by Buechler.

Finally, the classification of the primitive geometries associated with strongly minimal \(\aleph_0\)-categorical structures yields a classification of the structures themselves if singletons are algebraically closed (the primitive case). In the imprimitive case, the Galois group of the underlying field may act on the blocks of imprimitivity causing complications which are not understood, but is it conjectured that strongly minimal \(\aleph_0\)-categorical structures in which the Galois group does not act (i.e. field elements are named) should be classifiable. (This is true when the quotient is degenerate.)

References

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Notes

1. Zoe Chatzidakis has shown that profinite groups with the Iwasawa ("embedding") property are naturally associated with $\mathcal{S}_0$-categorical $\mathcal{S}_0$-stable many-sorted structures, to which a slight generalization of the theory described here applies.

2. The correction to [13] referred to above has been published in:


3. The conjecture in [4] has been proved in the manner anticipated.

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