1. Background: the \( \beta \)-rule

1.1. Among the contributions of Poland to mathematical logic, one of the most original and far-reaching is the theory of so-called \( \beta \)-models, introduced by Mostowski in the early 60's: A \( \beta \)-model of the second order arithmetic is a model where well-foundedness is absolute, i.e.

\[ M \models WF[\bar{\theta}] \leftrightarrow \{ (n, m) ; M \models \bar{n} \bar{m} \text{ is well-founded} \}. \]

This notion (together with the concept of \( \omega \)-model) is one of the genuine improvements of the concept of model, closely related to the concept of transitive model in set theory. A famous result of Mostowski and Suzuki states that we cannot by any means reduce \( \beta \)-models to \( \omega \)-models.

1.2. In analogy to the well-known completeness theorem of Orey for \( \omega \)-logic, Mostowski raised the question of characterizing validity in all \( \beta \)-models by a system of axioms and rules: the \( \beta \)-rule. The answer is by no means trivial: truth in all models is \( \Sigma_1 \)-complete, truth in all \( \omega \)-models is \( \Pi_1 \)-complete and truth in all \( \beta \)-models is \( \Pi_1 \)-complete; hence we must find, if we want to solve Mostowski's problem, a notion of proof drastically different from the finitary trees and the well-founded \( \omega \)-branching trees which answer the completeness problem for the usual models and the \( \omega \)-models respectively. In the middle 70's, Apt was able to show that no solution is possible without introducing a completely new idea.

1.3. A more general notion of \( \beta \)-model is the following: if \( T \) is a denumerable theory in a language \( L \) containing a distinguished type 0 for
ordinals, and a binary relation $<$ on that type, a $\beta$-model of $\mathcal{F}$ is a model of $\mathcal{F}$ such that $|M| = (M(0), M(<))$ is an ordinal. ($M(0) = \emptyset$ is allowed.) Mostowski’s notion is easily reduced to the new one, so we shall now only deal with that one. Assume that the formula $A$ is true in all the $\beta$-models of $\mathcal{F}$ such that $|M| = \alpha$. It follows immediately (at least when $\alpha$ is denumerable) that $A$ has an $\omega$-proof in $\mathcal{F}$, i.e. a proof making use of the $\omega$-rule:

$$
\Gamma \vdash A[\bar{x}], A \ldots \Gamma \vdash A[\bar{z}], A \ldots \forall z < \omega
$$

So, the truth of $A$ in all $\beta$-models of $\mathcal{F}$ implies the existence of a family $(\pi_\alpha)_{\alpha \in \omega}$ such that for all ordinals $\alpha$, $\pi_\alpha$ is an $\omega$-proof of $A$ in $\mathcal{F}$. Such a family can be considered as a $\beta$-proof of $A$ in $\mathcal{F}$. However, as it stands, our solution is both trivial and ridiculous: a proof must be a “syntactic” object, i.e., it must be finitarily graspable in some way! A family of infinitary proofs, indexed by the class of all ordinals (or even by denumerable ones) is by no means graspable. But the solution would not be ridiculous if we were able to generate our family $(\pi_\alpha)$ from a reasonable denumerable set of data, and to do this in an effective way!

1.4. If $\alpha, \alpha' \in \omega$ and $f \in I(\alpha, \alpha')$ (i.e., $f$ is a strictly increasing function from $\alpha$ to $\alpha'$), if $\pi$ is an $\alpha'$-proof, then one can (in certain cases) define an inverse image $f^{-1}(\pi)$ of $\pi$ under $f$; $f^{-1}(\pi)$ will be an $\omega$-proof. This process is called mutilation and runs in two steps:

(i) In $\pi$ delete all sequents occurring above any premise of any index $z \notin \text{rg}(f)$ of any $\omega$-rule.

(ii) If in the remaining “proof” all ordinals which occur are in $\text{rg}(f)$, then replace systematically parameters $f(z)$ by $\bar{z}$.

A $\beta$-proof of $A$ in $\mathcal{F}$ is a family $(\pi_\alpha)_{\alpha \in \omega}$ such that:

(i) For all $\alpha, \pi_\alpha$ is an $\omega$-proof of $A$ in $\mathcal{F}$,

(ii) For all $\alpha, \alpha'$ and $f \in I(\alpha, \alpha') f^{-1}(\pi_\alpha) = \pi_\alpha$.

An alternative formulation is to present a $\beta$-proof as a functor from the category $\text{ON}$ of ordinals into a category $\text{DEM}_\mathcal{F}$ of ordinal-branching proofs. Then one easily checks that such a functor preserves direct limits and pull-backs; and, since any ordinal is a direct limit of integers, the subfamily $(\pi_n)_{n < \omega}$ determines the complete family $(\pi_\alpha)_{\alpha \in \omega}$ and does this

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1 But given any family $(\pi_\alpha)_{\alpha \in \omega}$ such that $f \in I(n, m) \mapsto f^{-1}(\pi_m) = \pi_n$, the extensions $(\pi_\alpha)_{\alpha \in \omega}$ need not be well-founded for all $\alpha \in \omega$; similarly an analytic function defined on a neighbourhood of $\omega$ may have no extension to the disc $|z| < 1$ (but the extension, if it exists, is uniquely and effectively determined).
in a completely effective way. Since the proofs $\pi_n$ are obviously finite, it makes sense to speak of a recursive family $(\pi_n)_{n \in \omega}$: we can therefore define the concept of a recursive $\beta$-proof, and we have obtained a reasonable syntactic notion.

1.5. I introduced the concepts of 1.4 in 1978 and proved completeness: $A$ is true in all $\beta$-models of $\mathcal{F}$ iff there is a $\beta$-proof of $A$ in $\mathcal{F}$. Furthermore, when $\mathcal{F}$ is recursive, the $\beta$-proof can be chosen recursive. The result gave a complete solution to Mostowski's problem, and had a lot of applications in generalized recursion. The reason for these applications is quite simple: in many situations of admissible set-theory, the structure we are dealing with is the only $\beta$-model of a finite theory $\mathcal{F}$; combining completeness with cut-elimination techniques, I was able to reduce generalized recursion over reasonable successor-admissible ordinals to usual recursion. These results were simplified and/or improved by Masseron, Ressayre, Van de Wiele, Normann, Vauzeilles, Jäger, Buchholz,...

1.6. Because of its interesting semantics and its nice syntax, $\beta$-logic is the most natural generalization of $\omega$-logic. We are dealing with large objects (the ordinals $\omega$ in $\pi_\omega$ are arbitrarily large), but we keep a finitary control (by means of the $\pi_n$'s). Among the important formal properties of $\beta$-logic, let us mention:

(i) Interpolation: an interpolation theorem was obtained by Vauzeilles; similarly to $\omega$-logic, we must enlarge the language: it is necessary to introduce $L_{\beta_0}$ where the formulas themselves are functors; in other terms, we allow conjunctions of variable lengths.

(ii) Cut-elimination: $L_{\beta_0}$ enjoys a reasonable form of cut-elimination. Ferbus obtained bounds for the cut-elimination process by means of the functorial version of the Veblen hierarchy.

(iii) The underlying structure behind $\beta$-proofs, when endowed with the familiar Kleene–Brouwer ordering, leads to the concept of a dilator, fundamental in $\Pi^1_2$-logic, i.e., in the theory of all notions related to $\beta$-proofs and their applications. There is a close connection between syntactic operations on $\beta$-proofs and the natural properties of dilators.

1.7. However, $\beta$-logic, as it stands, is not completely satisfactory; in particular, we would like to make a $\beta$-proof appear as a (well-founded) succession of rules, the usual ones plus a specific one, the $\beta$-rule. The need for a specific $\beta$-rule arises from the fact that the most advanced techniques on the subject consist in cut-elimination theorems for theories of inductive...
definitions, in which a presentation of the deductive framework by means of axioms and rules is simply crucial. It is possible to find such a presentation for $\beta$-logic (see [1] and also the unpublished Ch. 2 of Vauzeilles Thèse d'État); but the solution thus found is not completely satisfactory.

1.8. $\Omega$-logic is the last word (?) on Mostowski's problem: by replacing the category ON by the wider WF, we obtain a greater flexibility (for instance WF has direct sums), and in particular a very satisfactory specific $\Omega$-rule. (We have chosen "$\Omega$" to stress both the analogy with and the difference from the case of $\omega$-logic.) The theory of the underlying structures behind $\Omega$-proofs is very rich: these objects are called gerbes and seem to be an adequate framework for a general geometry of ordinals.

2. The languages $L_{R,a}$

2.1. In the rest of the paper, $L$ is a fixed first order language; we shall save space by assuming that the formulas of $L$ are built from atomic ones by means of the unique operator $QxAB$ (whose intended meaning is $Vx(\sim A \& \sim B)$. The usual connectives and quantifiers are easily defined from $Q$.

2.2. WF is the category of well-founded orders:

$R \in |WF|$ iff $R$ is a binary relation (defined on a set $|B|$) which is irreflexive, transitive, and well-founded: there is no sequence $(a_n)_{n \in \mathbb{N}}$ in $|R|$ such that $a_{n+1} R a_n$ for all $n \in \mathbb{N}$.

$\varrho \in WF(R, S)$ iff $\varrho$ is a function from $|R|$ to $|S|$ such that $aRb \rightarrow \varrho(a) S \varrho(b)$. FIN is the full subcategory of WF consisting of finite orders.

2.3. A direct system$^3$ $(R_i, \varrho_{i\alpha})$ admits $(R, \varrho)$ as its direct limit in WF iff:

(i) $\varrho_i \in WF(R_i, R)$,

(ii) $i < \kappa \rightarrow \varrho_i = \varrho_{i+1}$,

(iii) If $a \in |R|$, then $a \in rg(\varrho_i)$ for some index $i$,

(iv) If $aRb$ there are some index $i$ and some $a', b' \in |R_i|$ s.t. $a = \varrho_i(a')$, $b = \varrho_i(b')$ and $a'R_i b'$. From this we see that every object of WF is the direct limit of a direct system in FIN. But most of direct systems in FIN have no direct limit in WF. In order to get a category containing WF and closed under direct

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$^2$ The "$\Omega$-rule" considered here has no known relation to the rule of the same name considered by Buchholz.

$^3$ A "direct system" means an inductive system indexed by a nonvoid directed ordered set.
limits, one must introduce the category OR of strict orders, i.e., drop
the well-foundedness condition in 2.2.

2.4. (i) If $R \in \mathcal{WF}$, we define $R+1$ by $|R+1| = |R| \cup \{ \bar{R} \}$; $a R+1 b$ iff $a R b$
when $a, b \in R$; $a R+1 \bar{R}$ when $a \in R$. When $\varphi \in \mathcal{WF}(R, S)$, we define
$\varphi+1 \in \mathcal{WF}(R+1, S+1)$ by $\varphi+1(a) = \varphi(a)$ for $a \in |R|$, $\varphi+1(\bar{R}) = S$.

(ii) If $R \in \mathcal{WF}$ and $i \in |R|$, we define $R^\ast i$ by $|R^\ast i| = |R| \cup \{ j \}$ (with
$j \notin |R|$) $a R^\ast ib$ iff $a R b$ when $a \in |R|$, $b \in |R|$, $j R^\ast ib$ iff $i = a$ or $i R a$. If
$\varphi \in \mathcal{WF}(R, S)$, if $i \in |R|$ and $b S \varphi(i)$, we define $\varphi^\ast ib \in \mathcal{WF}(R^\ast i, S)$ by:

$\varphi^\ast ib(a) = \varphi(a)$ for $a \in |R|$, $\varphi^\ast ib(j) = b$.

2.5. If $K \in \mathcal{WF}$, we define the category $\mathcal{WF}^K$ by:

$|\mathcal{WF}^K| = \{ (R, d); d \in \mathcal{WF}(K, R) \}$,

$\mathcal{WF}^K(R, d; S, e) = \{ \varphi \in \mathcal{WF}(R, S); e = \varphi d \}$.

2.6. Assume that $(R, d) \in |\mathcal{WF}^K|$; we then define the language $L_{R, d}$ inductively:

(i) If $A$ is an atomic formula of $L$, then $A \in L_{R, d}$,

(ii) If $A, B \in L_{R, d}$ and $x$ is a variable of $L$, then $Q x A B \in L_{R, d}$,

(iii) If $i \in |K|$, assume that $(A_b)_{b \in d(i)}$ is a family of formulas, $A_b \in L_{R, d_b}$,
such that only finitely many variables are free in the whole family; then

$\prod_{b \in d(i)} A_b \in L_{R, d}$;

(iv) The only formulas of the languages $L_{R, d}$ are those given by (i)–(iii).

2.7. Assume that $\varphi \in \mathcal{WF}^K(R, d; R', d')$ and that $A' \in L_{R', d'}$; then we define
$\varphi^{-1}(A') \in L_{R, d}$ by mutilation:

(i) $\varphi^{-1}(A') = A'$ if $A'$ is atomic,

(ii) $\varphi^{-1}(Q x A' B') = Q x \varphi^{-1}(A') \varphi^{-1}(B')$,

(iii) $\varphi^{-1}(\prod_{b \in d(i)} A_b) = \prod_{b \in d(i)} \varphi^{-1}(A_b)$: here we remark that

$\varphi \in \mathcal{WF}^K \{ (R, d^\ast i b; R', d'^\ast i \varphi(b)) \}$.

2.8. Example. Assume that $F[x, y]$ is a formula of $L$; we introduce, for $R \in |\mathcal{WF}|$ and $a \in |R|$, the formulas:

$\mathcal{Acc}(R, a, F', x) = \mathcal{Acc}(R, b, F', y)$,

$\mathcal{Acc}(R, F') = \mathcal{Acc}(R, b, F', x)$. 

\[ Acc(R, a, F', x) = \mathbf{W}_{b \in a} \forall y \{ F[y, x] \rightarrow Acc(R, b, F', y) \}, \]

\[ Acc(R, F') = \mathbf{W}_{b \in a} \forall x \mathcal{Acc}(R, b, F', x). \]
When $F$ is an order and $R$ is a well-order, then $Acc(F, R)$ means that $F$ is a well-order and $\|F\| < \|R\|$. If $|K| = \{a\}$ and $d_a(a) = a$ then $Acc(R, a, F, x) \in I_{R,d}$. If $q \in WF(R, R')$, then

$$q^{-1}(Acc(R', q(a), F, x)) = Acc(R, a, F, x).$$

$Acc(R, F) \in \mathbb{L}_{R+1,0}$ and, if $q \in WF(R, R')$ then

$$q + 1^{-1}(Acc(R', F)) = Acc(R, F).$$

2.9. If $K \in |WF|$, the category $FOR^K$ is defined by:

$|FOR^K| = \{(R, d, A); (R, d) \in |WF^K| \text{ and } A \in I_{R,d}\}$,

$q \in FOR^K(R, d, A; R', d', A')$ iff $q \in WF^K(R, d; R', d')$ and $q^{-1}(A') = A$.

2.10. The functor $type$: $t(R, d, A) = R$, $t(q) = q$ preserves and reflects direct limits: this means that

$$(R, d, A; q_i) = \lim(R_i, d_i; A_i; q_{i\alpha}) \text{ iff } (R, q_i) = \lim(R_i, q_{i\alpha}).$$

But this functor does not create direct limits: the existence of $\lim(R_i, q_{i\alpha})$ in $WF$ is not enough to ensure the existence of $\lim(R_i, d_i; A_i; q_{i\alpha})$ in $FOR^K$.

However, the direct limit will exist in a wider category $\overline{FOR^K}$, defined as $FOR^K$, $I_{R,d}$ being replaced by $\overline{I}_{R,d}$: the formulas of $I_{R,d}$ can be viewed as trees; these trees are characterized by local conditions (on the branchings) and a global one, well-foundedness. $\overline{I}_{R,d}$ is obtained by keeping the local conditions and dropping well-foundedness; the elements of $\overline{I}_{R,d}$ are called preformulas.

2.11. The category $REP$ is defined by: $|REP| = |WF|$, $q \in REP(R, S)$ iff $q$ is replete, i.e., $q$ is surjective and, for all $a \in |R|$, $q$ maps $\{b; bRa\}$ onto $\{b'; b'SQ(a)\}$.

2.12. Let us denote by $\theta_A$ the tree associated with a formula $A \in I_{R,d}$; if $A = q^{-1}(A')$, then one easily builds a function $\theta_q \in WF(\theta_A, \theta_A')$; this function is injective when $q$ is injective, and is surjective when $q$ is replete.

2.13. Assume that $M$ is a structure for $L$. Then we easily define the notion of validity of a closed formula $A \in I_{R,d}$ in $M$, denoted by $M \models A$. One easily checks that, if $q \in REP(R, R')$ and $A = q^{-1}(A')$, then $M \models A$ iff $M \models A'$. This illustrates the extreme importance of replete morphisms.

2.14. Given $R \in |WF|$, there are two main replete morphisms connected with $R$:
(i) the canonical morphism from $R$ to $\|R\|$,
(ii) if $T_R$ denotes the tree of nonvoid descending sequences of $R$, ordered by the extension relation, the function $\varphi \in \text{REP}(T_R; R)$:
$$\varphi((x_0, \ldots, x_n)) = x_n.$$ 

In particular, in the functional languages $L_{a,s,a}$, the meaning of $\Phi(R)$ will only depend on $\|R\|$, so $\Phi(.)$ only speaks about ordinals, and there is no real gain of expressive power w.r.t. $\beta$-logic; on the other hand, when $x$ is an ordinal, the formula $\Phi(T_x)$ has nicer geometrical properties than $\Phi(x)$...

3. The calculi $Lk_{R,s,a}$

3.1. In the sequel we shall work with the following stock of axioms and rules:

I. Axioms.
$$A \to A \quad (A \text{ atomic}).$$

II. Logical rules.

Q-introduction
$$\Gamma, A \to A, B \to \Pi \quad \rQ^4$$

Q-elimination
$$\Gamma \to A[t/x], A \quad \l1Q$$
$$\Gamma, Q[xAB] \to A \quad \l2Q$$

$\textbf{\Pi}$-introduction
$$\Gamma \to A_b, A \quad \r\Pi_{\text{b}}$$

$\textbf{\Pi}$-elimination
$$\Gamma, A_{b_a} \to A \quad \l\Pi_{\text{b}}$$

In $(\Pi\Pi_{\text{b}})$, $b_oRa$.

III. Structural rules. These rules are the usual ones: weakening $(rW)$ and $(lW)$, exchange $(rE)$ and $(lE)$, and contraction $(rC)$ and $(lC)$.

IV. Cut rule.
$$\Gamma \to A, A \quad \Pi \to A \quad \Pi \to A \quad \text{CUT}$$

3.2. Assume that $R \in \text{WF}$ and that $d \in \text{WF}(K, R+I)$; then we define the sequent calculus $Lk_{R,d}$; the sequents are expressions $\Gamma \to A$, where

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Footnote 4: $x$ is not free in $\Gamma$, $A \to A$, $\Pi$. 


$\Gamma$ and $\Delta$ are finite sequences of formulas in $L_{R+I,d}$. The proofs are defined inductively by iterating the rules and axioms given in 3.1; the rules for $\mathcal{M}$ deserve special mention:

(i) If for all $b$ such that $bR\bar{d}(i)$, $\pi_b$ is a proof of $\Gamma \rightarrow A_b$, $\Delta \in L_{k,R,d}$, then we can form a new proof $\pi$ of $\Gamma \rightarrow \bigwedge_{bR\bar{d}(i)} A_b$, $\Delta \in L_{k,R,d}$.

(ii) Assume that $\mathcal{M} \hat{\lambda}$; then from a proof $\lambda$ of $\Gamma, A_{d(\lambda)} \rightarrow \Delta$ in $L_{k,R,d}$ we can construct a proof $\pi$ of $\Gamma, \bigwedge_{bR\bar{d}(i)} A_b \rightarrow \Delta$ in $L_{k,R,d}$.

(iii) The remaining clauses are all of the following form: assume that $\pi_1, \ldots, \pi_k$ are proofs of $\Gamma_1 \rightarrow \Delta_1, \ldots, \Gamma_k \rightarrow \Delta_k$ in $L_{k,R,d}$ and that the rule or axiom (0) applies to these sequents, and yields $\Gamma \rightarrow \Delta$; then we can form a new proof $\pi$ of the sequent $\Gamma \rightarrow \Delta$ in $L_{k,R,d}$.

3.3. Assume $\varrho \in WF (R, R')$ and that $d' = \varrho + 1d$; then, with each proof $\pi'$ in $L_{k,R,d}$, we associate a proof

$$\pi = (\varrho, \sigma)^{-1}(\pi')$$

in $L_{k,R,d}$, on the model of 2.7, using the function $\varrho + 1$. If the conclusion of $\pi$ is $\Gamma' \rightarrow \Delta'$ ($= A'_1, \ldots, A'_n \rightarrow A'_{n+1}, \ldots, A'_m$), then the conclusion of $\pi$ is $\varrho^{-1}(\Gamma' \rightarrow \Delta')$ ($= A_1, \ldots, A_n \rightarrow A_{n+1}, \ldots, A_m$, with $A_i = \varrho + 1^{-1}(A'_i)$).

3.4. We can form a category $DEM^K$ of proofs. Its objects are 3-tuples $(R, d, \pi)$, with $\pi \in L_{k,R,d}$, and $\varrho \in DEM^K (R, d, \pi; R', d', \pi')$ iff $\varrho + 1 \in WF^K (R + 1, R + 1)$ and $\pi = (\varrho)^{-1}(\pi')$.

3.5. The functor "type": $\mathfrak{t}(R, d, \pi) = R, \mathfrak{t}(\varrho) = \varrho$, preserves and reflects direct limits; but it does not create them. It is easy to see that the analogues of Remarks 2.10 and 2.12 hold for $DEM^K$.

4. The calculi $L_{Q,S,d}$

4.1. Assume that $R, S \in |WF|$; then we define $R \oplus S$ by:

$$|R \oplus S| = \{o\} \times |R| \cup \{1\} \times |S|,$$

and

$$(i, a)R \oplus S(j, b) \text{ iff } i = j = o \text{ and } aRb \text{ or } i = j = 1 \text{ and } aSb.$$ 

If $\varrho \in WF (R, R')$, and if $\sigma \in WF (S, S')$, we define

$$\varrho \oplus \sigma \in WF (R \oplus S, R' \oplus S')$$
by
\[ e \oplus \sigma(o, a) = (o, e(a)), \quad e \oplus \sigma(1, a) = (1, e(a)). \]

Obviously, \( \oplus \) is a direct sum in the category WF.

We make the following notational conventions:

(i) We consider that \( \oplus \) is associative and that the void order \( \emptyset \) is neutral for \( \oplus \).

(ii) If \( \delta \in WF(K, S) \) and \( e \) is the only element of \( WF(\emptyset, R) \), then \((e \oplus \delta) + 1 \in WF((\emptyset \oplus K) + 1, (R \oplus S) + 1)\). By (i) we have \( \emptyset \oplus K = K \).

We shall use the notation \( \delta + 1 \) instead of \((e \oplus \delta) + 1\), hence
\[ \delta + 1 \in WF(K + 1, (R \oplus S) + 1). \]

4.2. Assume that \( S \in |WF| \) and \( \delta \in WF(K, S) \); then we define the language \( L_{\delta,S,\delta} \) as follows: the formulas are functors \( \Phi \) from WF to FOR\(^{\mathsf{K + I}} \) such that:

(i) \( \Phi(R) = ((R \oplus S) + 1, \delta + 1, \Phi_R) \),

(ii) \( \Phi(e) = (e \oplus \text{id}_S) + 1 \),

(iii) If \( \lambda_R \) is the canonical surjection from \((R \oplus S \oplus S) + 1\) to \((R \oplus S) + 1\), then \( \lambda^{-1}_R(\Phi_R) = \Phi_{R\oplus S} \) (Observe that \( \lambda_R \) is replete.)

4.3. The formulas of \( L_{\delta,S,\delta} \) preserve direct limits (they also preserve pull-backs and kernels). In particular, such functors are completely and effectively determined by the values \( \Phi(R) \) for \( R \in |\mathsf{FIN}| \). When \( S \) is itself finite, then these formulas are finite.

4.4. The calculus \( Lk_{\delta,S,\delta} \) is defined as follows: the proof are functors \( \Pi \) from WF to DEM\(^{\mathsf{K + I}} \) such that:

(i) \( \Pi(R) = (R \oplus S) + 1, \delta + 1, \Pi_R) \),

(ii) \( \Pi(e) = (e \oplus \text{id}_S) + 1 \),

(iii) \( \lambda^{-1}_R(\Pi(R)) = \Pi(R \oplus S) \)

4.5. For proofs in \( Lk_{\delta,S,\delta} \), remarks analogous to 4.3 are obviously true.

4.6. If the conclusion of \( \Pi \) is \( A^1_R, \ldots, A^n_R \Rightarrow A^{n+1}_R, \ldots, A^m_R \) then one can say that the conclusion of \( \Pi \) is the sequent \( \Phi^1, \ldots, \Phi^n \Rightarrow \Phi^{n+1}, \ldots, \Phi^m \) with \( \Phi^i_R = A^i_R \). One easily checks that the \( \Phi^i \)s are formulas of \( L_{\delta,S,\delta} \).

4.7. A structure for the language \( L_{\delta,S,\delta} \) is a pair \((M, R)\), where \( M \) is a structure for \( L \) and \( R \in |WF| \); if \( \Phi \) is a closed formula of \( L_{\delta,S,\delta} \), then \((M, R) \vdash \Phi \) means that \( M \vdash \Phi_R \).
4.8. Let $\mathcal{S}$ be a denumerable set of closed formulas of $L_{\alpha,S,n}$ and let $(M, R)$ be a structure for $L_{\alpha,S,n}$; then $(M, R)$ is a model of $\mathcal{S}$ iff all formulas of $\mathcal{S}$ are true in $(M, R)$. Proofs in $\mathcal{T} + Lk_{\alpha,S,n}$ are defined exactly as in 4.4, except that proper axioms $\Phi_R (\Phi \in \mathcal{T})$ are allowed in $\Pi_R$.

4.9. **Soundness Property.** Assume that the closed sequent $\vdash \Phi$ has a proof in $\mathcal{T} + Lk_{\alpha,S,n}$ and that $(M, R)$ is a model of $\mathcal{T}$; then $(M, R) \models \Phi$.

4.10. **Completeness Theorem.** Assume that $K$ is denumerable and that $d \in \text{REP}(K, S)$; if the closed formula $\Phi$ holds in all models of $\mathcal{T} + Lk_{\alpha,S,n}$, then it has a proof in $\mathcal{T} + Lk_{\alpha,S,n}$; this proof is recursive in the data.

*Proof.* If $R \in |WF|$, we build a preproof $\Pi_R$ of $\vdash \Phi_R$ by starting with the conclusion, and going upwards from premise to premise; it is possible to arrange a strategy in such a way that all possible rules are tried infinitely often... Assume that we have obtained the “hypothesis” $\Gamma \vdash \Delta$, $n$ steps above the conclusion $\vdash \Phi_R$, and that this sequent belongs to $Lk_{(R \otimes S)^n} + I(a,d+1)$ with $d \in WF (K', R \oplus S)$, $|K'| - |K|$ finite. Among the possibilities to go upwards we have:

(i) The case of a rule $(r \land)$, typically when $\Delta = \bigwedge a \in [R \otimes S] A_a, \Delta'$. We then put just above $\Gamma \vdash \Delta$ all sequents $\Gamma \vdash A_a, \Delta$ with $a \in [R \oplus S]$, and these new “hypotheses” belong to $Lk_{(R \otimes S)^n} + I(a,d+1)$.

(ii) The case of a rule $(l \land)$, typically when $\Gamma = \Gamma', \bigwedge a \in [R \otimes S] A_a$. Assume that $|K| = \bigcup |K_p|$, with $\text{card}(K_p) = p$ and assume that $|K' - |K|) \cup |K_n|$ is equal to $\{\bar{i_1}, \ldots, \bar{i_k}\}$; then we can form a finite sequence of rules $(l \land)$ and $(l0)$ leading from $\Gamma, A_d(i_1), \ldots, A_d(i_k) \vdash \Delta$ to $\Gamma \vdash \Delta$.

(iii) The case of a cut: the cut formula must be taken from among subformulas of $T_R \cup \{A_R\}$ in the language $L_{(R \otimes S)^n} + I(a,d+1)$; these subformulas can be enumerated into a sequence $(B^k_R, \ldots, B^k_R, \ldots)$, and the enumeration is “functorial”, i.e., $(\epsilon \otimes \text{id}_S) + I^{-1}(B^k_R) = B^k_R$ when

$$\epsilon \otimes \text{id}_S \in WF_{K'}(R \oplus S, d'; R \oplus S, d';)$$

also $\lambda_{R \otimes S}(B^k_R) = B^k_{R \oplus S}$. Then, by a big family of cuts on $B^k_R, \ldots, B^k_R$, we can infer $\Gamma \vdash \Delta$ from $2^n$ sequents

$$\Gamma, B^k_R, \ldots, B^k_R \vdash B^k_R, \ldots, B^k_R, \Delta.$$
well-foundedness. Now, if for some $R \in |WF|$, $\Pi_R$ has an infinite branch $(\Gamma_n \models A_n)_{n \in \mathbb{N}}$, one easily shows that the set of terms occurring in these sequent, equipped with the truth definition: $M \models A$ when $A \in \bigcup \Gamma_n$, $M \models \neg A$ when $A \in \bigcup \neg A$ defines a model $M$ of $T_R + \neg \Phi_R'$, where $R'$ is the restriction of $R$ to the set of all $a \in |R|$ which occur in some of these sequents. So $(M, R') \models \neg \Phi$ and is a model of $\mathcal{F}$, contradiction. ■

5. The $\Omega$-rule

5.1. Our task is now to reformulate our results in a more natural language, where formulas are built from the atomic ones by means of connectives and quantifiers (and similarly proofs are obtained from the axioms by means of rules).

5.2. Let $\Phi$ be a formula of $L_{\alpha, S, d}$ with $d \in WF(K, S)$;

(i) If $\Phi_a$ is an atomic formula $A$, then $\Phi_R = A$ for all $R$, and we can therefore identify $\Phi$ with the atomic formula $A$.

(ii) If $\Phi_a$ begins with $Qx$, then $\Phi_R = Qx \psi_R \theta_R$ for some $(\psi_R)_{Re|WF}$ and $(\theta_R)_{Re|WF}$; it is immediate that $(\psi_R)$ and $(\theta_R)$ define formulas $\psi$ and $\theta$ of $L_{\alpha, S, d}$: so we shall represent $\Phi$ by the notation: $\Phi = Qx \psi \theta$.

(iii) If $\Phi_a$ begins with $Ja$ with $^i \alpha$, then $a \in K \alpha$, so we can write without excessive abuse of notations: $\Phi_R = \bigwedge_{a \alpha \in \alpha \beta} \psi_a$. For all $a \in |S|$ such that $a \alpha \beta \in (i)$, the family $(\psi_a)_{Re|WF}$ defines the formula $\psi_a$ of $L_{\alpha, S, a \beta}$: so we shall represent $\Phi$ by the notation: $\Phi = \bigwedge_{a \alpha \beta} \psi_a$.

(iv) If $\Phi_a$ begins with $Ja$, then one can write $\Phi_R = \bigwedge_{a \alpha \beta} \psi_a$ (with little abuse of notation); observe that $\psi_a \in L_{(\alpha \beta) + 1 \alpha \beta}$ and that when $\alpha \in WF(R, R')$ we have $(\alpha \oplus \Id) \alpha + 1^{-1} (\psi^{(\alpha \beta)}) = \psi^R_R$. Now, take any $R \in |WF|$ and define $\theta^R \in L_{\alpha, R + 1 \alpha \beta}$ by $\theta^R_{\alpha \beta} = \psi^{(\alpha \beta)}_{L(R \oplus 1)} (d')$ is the extension of $d$ to $K \oplus (j)$ defined by $d'(j) = \bar{R}$. Here we represent $\Phi$ by:

$\Phi = \bigvee R \in WF \theta^R$.

5.3. We have so far succeeded in associating with any formula $\Phi$ of $L_{\alpha, S, d}$ an expression of that formula from other formulas (which will be styled the subformulas of $\Phi$) and (generalized) connectives and quantifiers. (Observe that in the case 5.2(iv) the subformulas involved form a proper class: this means that the expressions found in 5.2 are by no means small; however, for obvious direct limit reasons, the family $(\Phi^R)_{Re|WF}$ is completely and effectively determined by its restriction to $|FIN|$.)
Two questions remain:

(Q1) Given the expressions found for $\Phi$ in 5.2, can we recover $\Phi$ (uniquity)?

(Q2) Do all similar expressions correspond to a formula (existence)?

(i) In the case 5.2(i): (Q1) obviously the atomic formula $A$ determines $\Phi$. (Q2) any atomic formula of $L$ leads to some $\Phi$.

(ii) In the case 5.2(ii): (Q1) from $\Phi_R = Qx\Psi_R \theta_R$ it is clear that we can recover $\Phi$ from $\Psi$ and $\theta$. (Q2) given any $\Psi$ and $\theta$ we can build $\Phi = Qx\Psi \theta$ using the equation $\Phi_R = Qx\Psi_R \theta_R$.

(iii) In the case 5.2(iii): (Q1) using $\Phi_R = \bigsqcup_{\text{asd}(i)} \Psi_{\alpha}^a_R$, one can recover $\Phi$ from the family $(\Psi_{\alpha}^a)$. (Q2) Given such a family $(\Psi_{\alpha}^a)$, it is clear that we can define $\Phi$ by $\Phi_R = \bigsqcup_{\text{asd}(i)} \Psi_{\alpha}^a_R$.

(iv) In the case 5.2(iv): (Q1) we use the following property of replete morphisms: if $\varrho$ is replete, then $\varrho^{-1}(A)$ determines $A$ uniquely. Now take $a \in [R \oplus S]$; if $a \in [R]$, define $R_a$ by restricting $R$ to $\{b; b Ra\}$; then there is a replete morphism $\sigma \in \text{REPP}(R \oplus R_a + I, R)$ such that $\sigma(R_a) = a$, and so we get $\lambda^b_{\sigma} = (\sigma \oplus \text{id}_S) + I^{-1}(\Psi_R^a)$; if $a \in [S]$, define $\lambda^b_R$ as in 4.2(ii), and let $b$ be the element of $\lambda^b_R([a])$ in the first copy of $S$ (in $R \oplus S \oplus S$); then $\Psi^b_{R \oplus S} = \lambda^b_R(\Psi_R^a)$. (Q2) Given any family $(\varrho^R)_{R \in \mathcal{W}F}$ such that $\varrho^R \in L_{\mathcal{O}, R + 1 \in \mathcal{S}, a}$ there is no way of constructing a functor like $\Phi$ unless some inner solidarity is requested between the formulas $\varrho^R$. In general, when $\Lambda \in L_{\mathcal{O}, \mathcal{S}, \mathcal{A}}$ and $\varrho \in \mathcal{W}F(R_1, d_1; S_2, d_2)$ we can define $\varrho^{-1}(A)$ by

$$\varrho^{-1}(A)_R = (\text{id}_R \oplus \varrho) + I^{-1}(\Lambda_R).$$

The family $(\varrho^R)$ found in 5.2(iv) enjoys the solidarity condition:

(SOL) if $\varrho \in \mathcal{W}F(R, R')$, then $((\varrho + I) \oplus \text{id}_S)^{-1}(\varrho^R) = \varrho^R$.

It is not hard to see that given any family $(\varrho^R)$ enjoing this property, the answer just given to (Q.1) leads to a $\Phi$ such that $\Phi = V \in \mathcal{W}F \varrho^R$.

5.4. We have therefore obtained an alternative description of the language $L_{\mathcal{O}, \mathcal{S}, \mathcal{A}}$ in terms of connectives and quantifiers. In fact, this description would be free from any category-theoretic considerations if we were able to formulate the condition (SOL) not for formulas viewed as functors, but for the analytic version of formulas, by means of connectives and quantifiers: this amounts to defining directly $\varrho^{-1}$ on analytic formulas, and this offers neither surprise nor difficulties. Observe that the sub-

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*“analytic” version, by opposition to the concepts of 4., which are “synthetic”.}
formula relation is well-founded, but that the subformulas of a given
formula form in general a proper class. We can therefore perform induction
on the subformula relation, such induction being much powerful than
usual transfinite induction.

5.5. Now we try to find an analytic version of proofs: assume therefore
that $\Pi$ is a proof in $\text{Lk}_{\alpha, \kappa, \delta}$.

(i) If $\Pi_0$ is an axiom $A \rightarrow A$ or $\rightarrow B$ ($B \in T$), then $\Pi$ can be repre-
sented by the axiom $A \rightarrow A$ or the axiom $\rightarrow B$.

If $\Pi_0$ is not an axiom, then (0) be the name of the last rule of $\Pi_0$; then,
for any $R \in \{\text{WF}\}$, the conclusion $I_R \rightarrow A_R$ of $\Pi_R$ follows by means of the
same rule (0) from a family of premises $A'_R \rightarrow E'_R (i \in I_R)$, each of these
premises being itself proved by a subproof $\Sigma'_R$.

(ii) $(0) = (lQ), (l2Q), (rQ), (rO), (rW), (lW), (rE), (lE)$; for instance
$(0) = (l2Q)$. Then $I_R = \{o\}, I'_R = I_R, QxR, \delta R$ and $A'_R \rightarrow E'_R$ is

$$I'_R \rightarrow \theta R [t/\beta], A_R;$$

obviously we can say that $\Pi$ has been obtained by means of the rule

$$\frac{I', \theta \mid [t/\beta], A}{I', QxR, \delta R \rightarrow A} \text{ lQ}$$

applied to $\Sigma^o, \Sigma^o$ being the proof in $\text{Lk}_{\alpha, \kappa, \delta}$ corresponding to the family
$(\Sigma^o_{\alpha, \kappa, \delta, \delta})_{0, \alpha, \kappa, \delta}$.

(iii) $(0) = (rQ), (\text{CUT})$; then $I_R = \{o, I\}$ and, if one defines proofs
$\Sigma^o, \Sigma^l$ in $\text{Lk}_{\alpha, \kappa, \delta}$ by the families $(\Sigma^o_{\alpha, \kappa, \delta}), (\Sigma^l_{\alpha, \kappa, \delta})$, one can say that $\Pi$ is obtained
by means of

$$A_0, \Psi \rightarrow E_0, A'_0, \theta \rightarrow E'_0, \text{ rQ} \quad \text{or} \quad A_0 \rightarrow \Phi, \varepsilon_0, A'\varepsilon_0, \Phi \rightarrow E_1, \text{ CUT}$$

$$A_0, A'_0 \rightarrow QxR, \varepsilon_0, E_0, E'_0 \text{ or } \frac{A_0 \rightarrow \Phi, \varepsilon_0, A'\varepsilon_0, \Phi \rightarrow E'_1}{A_0, A'_0 \rightarrow E'_0, E'_1} \text{ or }$$

applied to $\Sigma^o$ and $\Sigma^l$.

(iv) $(0) = (rM)$ and $\Delta = a_{\alpha, \kappa, \delta}(i) \Psi^a, A'$. Then $I_R = \{b; bS(i)\}$. If $b \in I_R$
then $A'_R = I_R, E'_R = \Psi^b, A_R'$; if we define $\Sigma^b$ by the family $(\Sigma^b_R)$, then
$\Sigma^b$ is a proof of $I' \rightarrow \Psi^b, A'$, and we can apply the rule

$$\ldots I' \rightarrow \Psi^b, A' \ldots \text{ all } bS(i) \rightarrow \Psi^b, A' \text{ rM}$$

applied to the family of proofs $(\Sigma^b_{\alpha, \kappa, \delta})_b = \text{Lk}_{\alpha, \kappa, \delta}$ with $\Sigma^b \in Lk_{\alpha, \kappa, \delta}$, to obtain $\Pi$. \"
(v) \((\theta) = (l\Lambda)\) and \(I' = I', \bigwedge_{a \in S(d)} \Psi^a;\) then \(I_R = \{o\},\) and
\[
A_R^o = I'_R, \Psi^a_R
\]
\[
E_R^o = A_R. \text{ If we define a proof } \Sigma^o \text{ in } Lk_{Q, S, d} \text{ of } I', \Psi^a |\rightarrow \Delta \text{ by the family } (\Sigma^o_R), \text{ then it is clear that the rule}
\]
\[
\frac{I', \Psi^a |\rightarrow \Delta}{I', \bigwedge_{a \in S(d)} \Psi^a |\rightarrow \Delta} l\Lambda
\]
enables us to pass from \(\Sigma^o\) to \(\Pi.\)

(vi) \((\theta) = (r\Lambda)\) and \(\Delta = \forall R \in WF \theta_R, \Delta';\) then the indexing set \(I_R\) equals \(|R \oplus S|\) and, for \(a \in |R|, A_R = I'_R\) and \(E_R^a = \Psi^a_R, A'_R.\) Now define \(Z_R\) by
\[
Z_R^R = \Sigma_R^R \oplus \ell_R + 1; \text{ it is immediate that } Z_R^R \text{ is a proof of } I' |\rightarrow \theta_R, \Delta' \text{ in } Lk_{Q, R + \ell_S, d},
\]
and so we can say that \(\Pi\) is obtained from the family \((Z^R)_{\forall \in WF}\) by applying the rule
\[
\frac{\ldots \; I' |\rightarrow \theta_R, \Delta' \; \ldots \; all \; R \in |WF|}{I' |\rightarrow \forall R \in WF \theta_R, \Delta} l\Omega
\]
This rule is also called the \(\Omega\)-rule.

(vii) \((\theta) = (l\Lambda)\) and \(I' = I', \forall R \in WF \theta_R;\) then \(I_R = \{o\}\) and
\[
A_R^o = I'_R, \Psi^a_R
\]
\[
E_R^o = A_R. \text{ We would like to make } \Psi^a_R \text{ appear as something in terms of the family (functor!) } (\theta_R)_{\forall \in WF}. \text{ First observe that } d(i) \in |S|. \text{ Given any } a \in |S|, \text{ we can form } \theta^a, \text{ which is } \theta^S_a, \text{ with } S_a = S\{b \mid b \in S\}. \theta^a \text{ is a formula in } L_{Q, S_a + \ell_S, d} \text{ and if } \ell_a \text{ is the canonical morphism from } S_a + \ell_S \text{ to } S (\ell_a \text{ is replete), then the unique solution } \phi^a \text{ of } \ell_a^{-1}(\phi^a) = \theta^a \text{ is denoted by } (\theta^R)_{\forall \in WF}\{a\}. \text{ It is clear that}
\]
\[
\phi^a_R = \Psi^a_R,
\]
as a consequence of 4.2(iii). Hence we can obtain \(\Pi\) from the proof \(\Sigma^o\) defined by the family \((\Sigma^o_R)\) by means of the rule
\[
\frac{I', (\theta)_{\forall \in WF}[d(i)] |\rightarrow \Delta}{I', \forall R \in WF \theta_R |\rightarrow \Delta} l\Omega.
\]

5.6. If we now ask the questions of unicity and existence for the analytic proofs in a way similar to 5.3, then we obtain an answer similar to the one found for analytic formulas. In particular, a solidarity condition,
(SOL) if $e \in WF(R, R')$, then $((e + I) \oplus \operatorname{id}_S)^{-1}(Z^{R'}) = Z^R$,
is requested in case (vi).

5.7. Finally, the $\Omega$-rule is very close to the quantifier rules of usual logic: $(r\Omega)$ behaves like a rule $(r\mathcal{V})$, the functorial dependence being the generalization of the dependence upon a variable. More precisely, we see that eigenvariables are replaced in the context of the $\Omega$-rule by the arguments $R$ in $\lfloor WF \rfloor$ upon which our proof functorially depends.

$(l\Omega)$ behaves like a rule $(l\mathcal{V})$: we have defined the notion of substitution of a "constant" (i.e., substitute $S_a$ for the eigenvariable $R$), namely when we form $(\theta^R)_{\lfloor \text{WF} \rfloor}[a]$. The change of $S$ in passing from a proof to a subproof simply reflects the fact that some eigenvariables of $\Pi$ become constants in the subproof, similarly to the fact that a variable not used as an eigenvariable (i.e., not "closed") is a constant. It would be notationally convenient to represent families $(\theta^R)_{\lfloor \text{WF} \rfloor}$ enjoying solidarity by the notation $\theta[\xi]$, $\xi$ being a symbol for a "variable of type WF"...

5.8. If we look at the underlying structure behind formulas and proofs, we see that these structures are trees depending functorially on $R, e$ in $\lfloor WF \rfloor$. Such trees, equipped with the familiar Kleene–Brouwer ordering, turn out to be gerbes, a concept which is central to the geometry of ordinals. The most typical property of gerbes is the preservation of repleteness, which has deep consequences. A typical use of gerbes in connection with $\Omega$-logic would be to add function symbols, i.e., instead of having variables and constants (see 5.7) to have terms which are gerbes of their "variables".

5.9. With the usual $\beta$-rule, I obtained in 1979 the first full cut-elimination result for inductive definitions, with the applications mentioned in 1.5. The method goes through the iterated case, but stumbles on the case of the first recursively inaccessible ordinal, essentially because of the awkwardness of the analytic $\beta$-rule. It is likely that the $\Omega$-rule will provide an adequate framework for a cut-elimination for the first recursively inaccessible ordinal. The treatment of $\Pi^1_2$ comprehension stumbles on the same question (the other principles seeming quite clear), and so the $\Omega$-rule may have a large number of applications in the near future.

References

