The purpose of this report is to draw attention to a fruitful geometric method of the infinite group theory which has led the author to solving a number of group-theoretic problems. Among these are: the problem of O. Yu. Schmidt about the existence of an infinite non-abelian group in which all proper subgroups are finite \([8]-[10]\), the problem on the existence of a non-amenable group without free subgroups \([12]\) attributed to John von Neumann \([2]\) and the construction of the 'Tarski-monster' \([14]\). We will also mention a new relatively short proof of a theorem due to Novikov-Adyan \([13]\) and some other results.

The progress of topological investigations was one of the main stimuli for studying groups given by generators and defining relations. First connections of questions of the combinatorial topology with group-combinatorial ones were established by H. Poincaré, M. Dehn and other mathematicians on the eve of this century. First problems of the combinatorial group theory originate in geometry and topology.

Traditionally, many algebraic systems arose in connection with classification of geometric objects. But the situation when topological ideas help to answer a question in algebra is rare. Therefore, the simple but important observation of van Kampen \([18]\) has perhaps been overlooked by many authors. The essence of the van Kampen lemma is a clear geometric interpretation of deducibility consequences of defining relations. Before we pass to precise definitions we illustrate a simple case; for example, the equation

\[ a^2 ba^{-4} b^2 ab^{-1} a^{-1} b^{-1} a^{-1} b^{-1} = 1 \]

follows from relations \(a^3 = 1\) and \(aba^{-1}b^{-1} = 1\), i.e., \(ab = ba\) (Fig. 1).

We read one of the defining relations by going around any region and
we read a consequence by going around the boundary of the map (passing
the inverse edge gives the inverse letter).

The universal character of the van Kampen lemma makes the geometric
approach to studying group presentation very natural. This approach
yields an attraction of elementary facts in the combinatorial topology
(first of all, the Euler formula and the Jordan lemma) for solving some
problems arising in algebra. In the middle of the sixties the van Kampen
lemma was discovered and successfully applied by a number of mathema-
ticians to the Small Cancellation Theory, the central ideas of which are

Following [5], we give some definitions. Let $\mathbb{R}^2$ denote the Euclidean
plane. For a subset $S \subseteq \mathbb{R}^2$ we denote the boundary of $S$ by $\partial S$, and
the closure of $S$ by $\bar{S}$. A map $M$ is a finite collection of vertices (points of $\mathbb{R}^2$),
edges (bounded subsets homeomorphic to the open unit interval) and
cells (bounded subsets homeomorphic to the open unit disk) that are
pairwise disjoint and satisfy the following conditions: (i) if $e$ is an edge
of $M$, then there exist vertices $a$ and $b$ such that $\bar{e} = e \cup \{a\} \cup \{b\}$, (ii) the
boundary $\partial \Pi$ of each cell $\Pi$ of $M$ is $\bar{e}_1 \cup \bar{e}_2 \cup \ldots \cup \bar{e}_n$ for some edges $e_1, \ldots, e_n$
of $M$. Edges of maps will be viewed as having two possible orientations.

Let $\{a_1, a_2, \ldots\}$ be an alphabet and let a group $G$ be defined by its
presentation:

$$\langle a_1, a_2, \ldots | R_1 = 1, R_2 = 1, \ldots \rangle,$$

(1)

where $\{R_i\}_{i=1}^\infty = \mathcal{R}$ is the set of defining words of $G$. By a diagram $\Delta$ over
$\mathcal{R}$ we mean an oriented, connected, simply connected map $M$ and a function
$\varphi$ such that (i) the function $\varphi$ assigns to each oriented edge $e$ of $\Delta$ one of
the letters $a_1^{\pm 1}, a_2^{\pm 1}, \ldots$ (the label of $e$) and $\varphi(e^{-1}) = \varphi(e)^{-1}$, (ii) if $e_1 \ldots e_n$
is the boundary path of some cell $\Pi$ of $\Delta$, then $\varphi(e_1) \ldots \varphi(e_n) \mathcal{R}$ is
a cyclic shift of some $R_i^{\pm 1}$, $i = 1, 2, \ldots \ (\mathcal{R}$ denotes graphical equality).

The statement of the van Kampen lemma is almost obvious: a relation
$W = 1$ is a consequence of (1), iff there exist a diagram $\Delta$ over (1) with
the boundary path $e_1...e_m$ of $\Delta$ such that $\varphi(e_1)\varphi(e_2)...\varphi(e_m) \equiv W$ [5].
Moreover, one may assume that $\Delta$ is reduced, i.e., there is no pair of cells $\Pi_1, \Pi_2$ with a common edge $e \subseteq d\Pi_1 \cap d\Pi_2$ such that for boundary paths $e\varphi_1, e^{-1}\varphi_2^{-1}$ of $\Pi_1, \Pi_2$ the equation $\varphi(p_1) = \varphi(p_2)$ holds in the free group.

It is convenient to define $A^*$ as a set of cyclic shifts of words $R^{\pm 1} \in A$. A common initial segment of two different words of $A^*$ is called a piece. The most popular condition $C'(\lambda)$ of the Small Cancellation Theory says that if $U$ is a piece and $R \equiv UV \in A^*$, then $|U| < \lambda|R|$. Throughout this paper, a maximal subpath $p$ of the boundary path of a cell $\Pi$ such that $p^{-1}$ is a subpath of some cell $\Pi'$ or $p^{-1}$ is a subpath of the boundary path of $\Delta$ will be called an arc of $\Pi$. The number of arcs of $\Pi$ will be called a degree of $\Pi$. The condition $C'(\lambda)$ means that the degree of every interior cell of a van Kampen diagram is greater than $\lambda^{-1}$. If $\lambda$ is small, then Euler’s formula proves the existence of a cell $\Pi$ with a ‘long’ external arc $p$ (for instance, $|p| > \frac{1}{2}|d\Pi|$). It is clear that we have a basis for Dehn’s equality algorithm: a non-trivial cyclic reduced word $W$ equals 1 in $G$ iff $W$ contains a cyclic subword $U$ of some $R \in A^*$ and $|U| > \frac{3}{2}|R|$. M. Dehn applied the algorithm for the fundamental groups of compact Riemann surfaces of genus $g > 1$ (then the condition $C'(1/(4g-1))$ holds).

Many important and typical results of Small Cancellation Theory are exhibited in [5]. Contributions to this theory are made by V. A. Tartakovskyi, M. D. Greendlinger, J. L. Britton, H. Schiek, R. Lyndon, C. M. Weinbaum, P. E. Schupp, C. Lipschutz, A. I. Gol’berg and some others (see [5]).

Many important groups, however, have no presentation with $C'(\lambda)$ for small $\lambda$, or similar conditions. Firstly, the groups with small cancellation conditions contain free non-cyclic subgroups and many normal subgroups. Secondly, we have the following example. In order to construct a non-trivial finitely generated periodical group it is natural to try to define the set $A$ of defining words as follows. Let $R_1 \equiv a_1^{n_1}$, where $n_1$ is a sufficiently large number. Having defined $R_1, ..., R_{i-1} \in A$, let $A_i$ be the shortest word of infinite order in $G_{i-1} = \langle a_1, ..., a_m | R_1 = 1, ..., R_{i-1} = 1 \rangle$. We put $R_i \equiv A_i^{n_i}$. It is clear that the presentation gives a periodical group if $A = \bigcup_{i=1}^{\infty} \{R_i\}$. But it is impossible to use any $C'(\lambda)$ condition for proving the group being infinite. Indeed, if $R_j \equiv UV$ and $|U| \approx \frac{1}{2}|R_j|$, then it is possible that for some $i > j$, $A_i \equiv UV'$, i.e., $U$ is a piece.

We develop the above geometric method for constructing groups with completely new properties. Thus, we obtain answers to the questions
whose appearance was very natural in the period, when the group theory was passing from finite to infinite objects. Their solutions required the introduction of new concepts. We formulate at first the theorems concerning the existence of groups with a very simple subgroup structure.

A **Noetherian group** is a group with no infinite ascending chain of subgroups, i.e., each of its subgroup is finitely generated. The following question of Baer is well known: Does an arbitrary Noetherian group \( G \) possess a finite series \( \{1\} = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G \), where each factor \( G_k/G_{k-1} \) is a finite or a cyclic group? (The converse is obviously true.)

**Theorem 1** ([8],[9]). There exists an infinite simple Noetherian group \( G \). Moreover, (i) \( G \) is effectively defined and there exist equality and conjugacy algorithms for \( G \), (ii) each proper subgroup of \( G \) is infinite cyclic, (iii) extraction of roots in \( G \) is unique: \( X^n = Y^n (n \neq 0) \) implies \( X = Y \) for \( X, Y \in G \).

An **Artinian group** is a group without infinite descending chains of subgroups. A group containing a subgroup of finite index which is a direct product of finitely many quasicyclic groups is called a **Chernikov group**. It is obvious that any Chernikov group is Artinian. Is the converse true (see [3])?

**Theorem 2** ([8],[10]). There is an infinite group \( G \) in which every proper subgroup has a prime order and any two subgroups of the same order are conjugate. \( G \) can be effectively presented with two generators in such a way that there exist algorithms for recognizing both equality and conjugacy in \( G \).

Thus, of course, a positive answer to the problem of O. Yu. Schmidt is obtained (see [3]) about the existence of an infinite group in which all proper subgroups are finite and which is different from the quasicyclic groups. It disproves the conjecture [4] on the finiteness of a group with both increasing and decreasing chain conditions. It produces a negative answer to the question of A. G. Kurosh and S. N. Chernikov [4]: Does the minimum condition for subgroups imply the local finiteness of a group? The existence of uncountably many non-isomorphic groups like those in Theorems 1, 2 is proved in [11]. It also gives answers to some problems about the structure of a lattice of subgroups (see [16]) and to number of other questions raised in the group theory.

One of the features of author’s papers is the proof of a large number of lemmas by simultaneous induction on a natural parameter \( i \) which, following the example of [7], we call the rank. The construction of the groups results from successive addition of defining relations of new ranks.
The group $G$, whose construction is the goal, is defined by a certain infinite sequence of defining relations, $G = \langle a, b | R_i = 1, \ldots \rangle$, where the lengths $|R_i|$ increase very rapidly as $i \to \infty$.

For the definition of the word $R_i$ we choose [8], [9] the, in a sense, minimal pair of words $\{A, B\}$ in the alphabet $a^{\pm 1}, b^{\pm 1}$ such that $AB \neq BA$ in $G_{i-1} = \langle a, b | R_1 = 1, \ldots, R_{i-1} = 1 \rangle$, and the subgroup $\text{gp}\{A, B\}$ is proper in $G_{i-1}$. In accordance with $A < B$ or $B < A$ the word $R_i$ has the form (we simplify slightly the correct definition [8])

$$R_i \equiv aBA^{k_i,0} BA^{k_i,1} B \cdots A^{k_i,h_i} \quad \text{or} \quad R_i \equiv bBA^{k_i,0} BA^{k_i,1} B \cdots A^{k_i,h_i},$$

where $k_i, h_i$ are some rapidly increasing parameters. In [8], [10] the definition of the word $R_i$ is similar to (2) for even $i$. If $i$ is an odd number then $R_i \equiv C_i p_i$, where $C_i$ is the minimal, in a sense, word whose order in $G_{i-1}$ is infinite, and $p_i$ is a certain large prime number. It is not very difficult to show that every proper subgroup of $G$ is abelian (and that $G$ is a periodical group in Theorem 2). But why is $G$ a non-abelian or even a non-trivial group?

An important technical tool in proving this is using diagrams with long periodic words written on their boundaries. A periodic word with the period $C$ is a subword of a certain power $C^n$ of word $C$. We note that the study (without connections with combinatorial topology) of periodic words and relations was started by V. A. Tartakovskij in the forties. The powerful elaboration of new combinatorial ideas for the investigation of transformations of periodic words was obtained by P. S. Novikov and S. I. Adyan [6], [7] in solving the restricted Burnside problem for sufficiently large odd exponents $n, n \geq 665$ [1].

As a rule, we study periodic words with simple periods, called simple if they are cyclically irreducible and not proper powers in the free group. Two partitions $X \equiv X_1 \cdots X_n$ and $Y \equiv Y_1 \cdots Y_n$, where $|X|, |Y| \geq |A|$, of periodic words with a period $A$ will be called $A$-consistent if $X$ and $Y$ are the subwords of a certain power of $A$ such that $A^k \equiv C_1 XD_1 \equiv C_2 XD_2$ and the difference $|C_1 X_1| - |C_2 Y_1|$ is a multiple of $|A|$. The following properties of periodic words with simple periods are almost obvious:

(i) If $|X| \geq |A|$ and $A^k \equiv C_1 XD_1 \equiv C_2 XD_2$, then $C_1 XD_1$ and $C_2 XD_2$ are the $A$-consistent partitions.

(ii) If $|X| \geq |A| + |B|$ and $X$ is both $A$- and $B$-periodic, then $B$ is a cyclic shift of $A$.

(iii) If $X$ is both $A$- and $A^{-1}$-periodic word, then $|X| < |A|$.
As a matter of fact, these properties of periodic words in the free group can be transferred with a modification to any group $G_t$. Among the new geometric ideas an important role is played by the idea of a band, that is, a 'narrow' and 'long' diagram with periodic boundaries:

![Diagram of a band](image)

Fig. 2

Suppose $p_1 q_1 p_2 q_2$ is the boundary path of the reduced diagram $\Delta$ of rank $i$ (i.e. over the presentation $G_i$), $\varphi(q_1)$ is an $A$-periodic word, $\varphi(q_2)$ is a $B$-periodic word, and $|p_1|, |p_2|, |A|, |B| \ll |q_1|, |q_2|$, where $A$ and $B$ are simple in rank $i$, i.e., words which are minimal in their conjugacy classes of $G_i$ and are not proper powers in $G_i$. Then:

(i) If $A \nmid B^{-1}$, then the paths $q_1$ and $q_2$ have a common vertex $o$, which gives $A$-consistent partitions of $\varphi(q_1)$ and $\varphi(q_2)$.

(ii) $B$ is conjugate to $A^{-1}$ in $G_i$.

(iii) $A \neq B$ in $G_i$.

Our considerations of diagrams of rank $i$ (a lot of details are avoided here) show that either a certain long subword of some defining word occurs in the boundary label of $\Delta$ or there exist two cells $\Pi_1, \Pi_2$ of the same rank whose boundary paths have many common edges. In the second case there exist bands 'between' $\Pi_1$ and $\Pi_2$. But the periodic structure of defining relations and the above properties of bands lead to a conclusion that the pair $\Pi_1, \Pi_2$ does not satisfy the condition in the definition of the reduced diagram. Thus, as in the Small Cancellation Theory, one deduces the existence, in any cyclically reduced consequence of $R_1, \ldots, R_i$, of a long piece of rank $j \leq i$. This enables us to prove, for example, that $G_i$ is infinite, hence also $G$ is infinite.

Other properties of $G$ follow from a study of diagrams on a sphere or a torus. For instance, the nonexistence of reduced diagrams on a torus is used in the proof that any subgroup of $G$ generated by two commuting elements is cyclic. It is known also (P. E. Schupp, see [5]) that conjugacy, as well as equality, has a natural geometric interpretation (with the help of annular diagrams). For the inductive proof of Theorems 1, 2 it is convenient to consider diagrams with an arbitrary number of 'holes'. The main additional difficulty in the proof of Theorem 2 is connected with the search, made geometrically, of elements of infinite order in the nonabelian subgroups of $G_i$. 

Recently G. S. Deryabina has made some alterations in the proofs of Theorems 1, 2. An infinite nonabelian group with finite proper subgroups is called a Schmidt group. No 2-group is a Schmidt group (O. Yu. Schmidt [17]). However, for any prime \( p \geq 3 \), G. S. Deryabina has proved the existence of uncountably many non-isomorphic Schmidt \( p \)-groups with isomorphic lattices of subgroups (to appear). All the proper subgroups in these examples are cyclic and the maximal subgroups of the same order are conjugate.

There exist many equivalent definitions of the amenable group (see [2]). John von Neumann proved [19] that every locally solvable group is amenable and a group containing a free noncyclic subgroup is not amenable. The conjecture that any group without noncyclic free subgroups is amenable is attributed by Greenleaf [2] to John von Neumann [19]. We disprove it in [12].

**Theorem 3.** There exists a nonamenable group \( G \) such that every proper subgroup of \( G \) is cyclic.

The proof is based on geometric estimations of the growth of the numbers \( d_n \) of words \( W \) with \( |W| = n \) in the kernel of the presentation of Theorem 1 or 2.

As we have remarked above, the restricted Burnside problem for sufficiently large odd exponents has been solved in [7]. However, the considerable size and a complicated logical structure are the real obstacle to studying their paper. In [13] we succeeded in producing a new, relatively short, proof of the well-known theorem of Novikov--Adyan. It should be mentioned that our estimate for the exponent \( n \) is worse than that in [1].

**Theorem 4.** For any \( m > 1 \) and any \( n > 10^{10} \) there exists an infinite \( m \)-generator group of exponent \( n \) (i.e. with the law \( x^n = 1 \)).

A natural presentation of the given free Burnside group [13] can be described as follows. To define the \( i \)th relation we choose the shortest word \( C_i \), in the alphabet \( a_1^{\pm 1}, \ldots, a_m^{\pm 1} \), of infinite order in the group \( \langle a_1, \ldots, a_m | R_1 = 1, \ldots, R_{q-1} = 1 \rangle \). Then we put \( R_i \supseteq C_i \).

Unlike [9], [10] the growth of \( |R_i| \) cannot be arbitrarily rapid as \( i \to \infty \). Moreover, the law \( x^n = 1 \) causes the logarithmic growth of \( |R_i| \). Therefore we have to find more delicate methods of dealing with bands. In place of this notion some other notions have appeared in [13]. Thus, we introduce the inductive notion of the contact diagram and the notion of a smooth section. An important Lemma 4.2 [13] states that the length of a smooth
section of a path can be compared with another path between the same vertices.

The term 'Tarski-monster' appeared before the corresponding group was constructed in [14]. The problem lay in the existence of an infinite group in which every proper subgroup has the same prime order. The example [14] combines two strong finiteness conditions: it is the Schmidt group of restricted exponent.

**Theorem 5.** For every sufficiently large prime \( p > 10^{75} \) there exists an infinite group \( G \) such that every proper subgroup of \( G \) has order \( p \).

New obstacles in [14] concern proving the cyclicity of every proper subgroup. Avoiding the details here, we note only that in [8]–[12] it is easy to achieve \( |A_{k,j}| \geq |B| \) in relations (2) (even in the case \( |A| \leq |B| \)) by choosing \( k_{i,j} \) sufficiently large. But it is obvious that in [14] all parameters should be bounded. The proof of Theorem 5 is based only on [13], although we use some ideas of [8], [9].

As the traditional Small Cancellations Theory, the author’s method can be applied to relations additionally imposed on free products. For instance, starting from the group \( A \ast B \), where \( |B| = 3 \), G. S. Deryabina (unpublished) has proved that every finite group \( A \) of odd order can be embedded in a Schmidt group with a number of additional properties.

In papers [8]–[14] we construct aspherical presentations (although in [13], [14] our notion of asphericity is weaker than in [5], the latter being connected with stronger restrictions for the reducibility of diagrams). Using the asphericity of presentations under consideration, we prove that \( F[N]/N \)-module \( N/[N,N] \), \( F \) being a free group and \( N \) the kernel of presentation (1), has no nontrivial relations (I. S. Ashmanov, A. Yu. Ol’šanskiï, unpublished). This fact enables us to construct abelian, in particular central, extensions of the above groups. For example, replacing the relations \( R_1 = 1, R_2 = 1, \ldots \) of the group in Theorem 4 by \( R_1 = R_2 = \ldots \) leads to the known group \( A(m,n) \) due to S. I. Adyan, which answers a number of questions (see [1]). In particular, a factor-group of it is an answer to a problem of A. A. Markov on the existence of a countable nontopologized group [15]. A similar central extension of the group in Theorem 5 (now the centre is of order \( p \)) gives an answer to a known question of P. Hall (see, for instance, [21]): Is it true that the finiteness of a verbal subgroup \( V \) of a Noetherian group \( G \) implies finiteness of the index of the corresponding marginal subgroup \( V^* \)? The word that disproves the conjecture is \( \omega^p \) where \( p \) is the number in Theorem 5.

Thus, the new method has shown its power in constructing groups with
a priori unknown but desired properties. There is no doubt that this method will also be used for solving some other group-theoretic problems. Finally, we mention an interesting paper of E. Rips [20], in which the theory of defining relations with special conditions has been elaborated. These conditions generalize the usual small cancellation hypotheses. For presentations with these conditions, E. Rips solves the word problem. The author of [20] has promised to apply, in the future, his theory to constructing groups solving some particular group-theoretic problems.

The author is grateful to Yu. A. Bakhturin for discussions during the writing.

References


