Birational Geometry of Algebraic Varieties

This is a continuation of articles read by B. Moishezon (1974, Nice) and K. Ueno (1978, Helsinki). Our purpose is to present a perspective view on the recent progress of the birational classification theory of algebraic varieties.

For simplicity, we restrict ourselves to algebraic varieties defined over the field of complex numbers.

§ 1. Preliminary

Needless to say, the prototype of classification theory of varieties is the classical classification theory of algebraic surfaces by the Italian school, enriched by Zariski, Kodaira and others.

Let me start by recalling the basic notions. Given a variety $V$, we have a non-singular model by Hironaka; this implies that there exist a non-singular variety $V_1$ and a proper birational map $\mu: V_1 \to V$. By using this, we define the following basic birational invariants:

$$P_m(V) := \dim H^0(V_1, \omega^m)$$

for any $m > 0$, and

$$g(V) := \dim H^0(V_1, \Omega^1).$$

Here, $\Omega^1$ denotes the sheaf of regular differential 1-forms and $\omega^m$ the $m$-times tensor product of the sheaf of regular $n$-forms, $n$ being $\dim V$.

Then the Kodaira dimension of $V$ is defined to be the non-negative integer $\kappa$ satisfying the following estimate: whenever $P_{m_0}(V) > 0$ for some positive integer $m_0$,

$$\alpha m^\kappa \geq P_{mm_0}(V) \geq \beta m^\kappa.$$

Here $\alpha$ and $\beta$ are positive numbers and $m$ is sufficiently large. If $P_m(V) = 0$ for all $m$, the Kodaira dimension is defined to be $-\infty$. 

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The Kodaira dimension is a birational invariant which takes one of the values \(-\infty, 0, 1, \ldots, n\). This is the most basic birational invariant in this theory. The fundamental structures of \(V\) are revealed by studying their Kodaira dimension.

§ 2. Fundamental results

Let \(f: V \to W\) be an algebraic fiber space, i.e., \(f\) is a proper and surjective morphism with connected fibers. We denote the dimensions of \(V\) and \(W\) by \(n\) and \(m\), respectively.

The following facts are easily proved (cf. [5]):

**Easy Addition Theorem.** For an algebraic fiber space in the above sense, we have

\[
\kappa(V) \leq \kappa(F_w) + \dim W.
\]

Here, \(F_w\) is a general fiber = \(f^{-1}(w)\), \(w\) being a general point of \(W\).

Note that by the definition of Kodaira dimension if \(\kappa(F_w) = -\infty\), then \(\kappa(V) = -\infty\).

**Covering Lemma.** If \(V_1 \to V\) is an étale covering, then \(\kappa(V_1) = \kappa(V)\).

**Canonical Fibering Theorem.** If \(\kappa(V) \geq 0\), then there exist a non-singular variety \(V^*\), a birational map \(\mu: V^* \to V\) and a fiber space \(f: V^* \to W\) such that (1) \(\dim W = \kappa(V)\), (2) \(\kappa(F_w) = 0\) for general points \(w\) of \(W\). Moreover, such an algebraic fiber space \(f\) is unique up to birational equivalence.

Thanks to the canonical fibering theorem, the study of varieties \(V\) is reduced to that of \(V\) with \(\kappa(V) = -\infty\) or \(0\) or \(n\), \(n\) being \(\dim V\).

Unfortunately, the following fundamental conjecture has not yet been solved completely:

**Conjecture.** \(C_{n,m}: \kappa(V) \geq \kappa(F_w) + \kappa(W)\).

\(C_n\) means \(C_{n,m}\) for any \(m > 0\).

In the following cases, the conjecture is verified:

(1) \(W\) is a variety of general type, i.e., \(\kappa(W) = \dim W\) (by Kawamata [9] and Viehweg [17]).

(2) General fibers have trivial pluri-canonical divisors (by Kawamata [11], [12]).

(3) General fibers are curves or surfaces (by Viehweg and Kawamata, see [11], [12]).

**Varieties** \(V\) with \(\kappa(V) = -\infty\). In the case of surfaces, such \(V\) are ruled surfaces (Enriques). If \(n = 3\), \(\kappa(V) = -\infty\) and \(q(V) > 0\), then
there exists a (possibly ramified) covering \( h: V_1 \to V \) such that \( V_1 \) is ruled. This was proved by applying C\(_8\) (Viehweg [16]). In general if a variety \( V \) has a covering \( V_1 \) that is ruled, \( V \) is said to be a uniruled variety or a quasiruled variety. The collection of polarized Kaehler varieties which are not uniruled has nice algebraic space structures (A. Fujiki [3]). Recently a variety whose anticanonical divisor is ample has been proved to be uniruled. This result follows from Mori’s theory of cones of 1-cycles (see Mori’s article in this volume).

Varieties \( V \) with \( \kappa(V) = 0 \). Then the Albanese map \( \alpha: V \to \text{Alb}(V) \) is a fiber space (Kawamata [9]), i.e., \( \alpha \) is surjective and has connected fibers. In particular, if \( n = q(V) (= \dim \text{Alb}(V)) \), then \( \alpha \) is a birational morphism. This follows from C\(_n\) and it gives a birational characterization of abelian varieties.

A general theory of varieties \( V \) with \( \kappa(V) = 0 \) and \( q(V) = 0 \) has not yet been established. For surfaces, such \( V \) are birationally equivalent to K3 surfaces or Enriques surfaces. When \( V \) satisfies \( \kappa(V) = 0 \) and \( 0 < q(V) < n \), the Albanese fiber space \( V \to \text{Alb}(V) \) seems to be birationally equivalent to a fiber bundle whose general fibers are varieties \( F \) with \( \kappa(F) = 0 \). For 3-folds this is verified by Viehweg [16].

Varieties \( V \) with \( \kappa(V) = n \). It has not yet been established that the graded ring \( B(V) = \bigoplus_{i=0}^n H^0(V, \omega^i) \) is finitely generated. When \( n = 3 \) and a canonical divisor \( K_F \) is numerically semipositive (or numerically effective), i.e., \( K_F \cdot C \geq 0 \) for any curve \( C \), this was proved by Kawamata [13] and X. Benveniste [1]. They proved a stronger result to the effect that \( K_F \) is semiample in this case, i.e., there is a pluricanonical system \( |mK_F| \) which has no base points at all for some \( m > 0 \). This is closely related to the construction of good minimal models for threefolds.

§ 3. Birational geometry

Birational geometry is the study of birational equivalence classes of varieties. Since any variety is birationally equivalent to a complete non-singular one, it may be enough to study non-singular complete varieties. But when one investigates the structure of non-complete varieties, it is occasionally helpful to use strictly rational maps and proper birational maps.

Let us recall their definitions.
DEFINITION. A rational map \( f: V \rightarrow W \) is said to be a \textit{strictly rational map} if the projection from the graph to \( V \) is a proper birational morphism. Further, a birational map is said to be a \textit{proper birational map} if both \( f \) and \( f^{-1} \) are strictly rational maps.

Any rational map into a complete variety is strictly rational. If a variety \( V \) is normal and \( f: V \rightarrow W \) is a strictly rational map, then \( f \) is defined outside a closed subset \( F \) with \( \text{codim}(F) \geq 2 \). In particular, if \( W \) is affine and \( V \) is normal, every strictly rational map is a morphism.

When one uses strictly rational maps and studies proper birational equivalence classes, one has proper birational geometry. More generally, considering variations of "rational maps", "birational equivalence classes" and "non-singular models", we can develop various kinds of birational geometries.

An outline of such a birational geometry is given below:

1. Fix a subset \( \mathcal{M} \) of the set of varieties and define rational maps and birational equivalence classes in \( \mathcal{M} \).
2. Introduce non-singular objects in \( \mathcal{M} \) and prove the existence of non-singular models for any object in \( \mathcal{M} \).
3. Using non-singular models, define regular forms and introduce plurigenera and Kodaira dimension for any object in \( \mathcal{M} \).
4. Obtain a rough classification of objects in \( \mathcal{M} \) by means of the Kodaira dimension.
5. Study particular structures of objects with the Kodaira dimension \(- \infty \) or 0 or \( n \), \( n \) being the dimension.
6. Introduce the notion of minimal models and study further...

In any case the classical classification theory of algebraic surfaces is a good guide. Recall that the classification of higher dimensional varieties has been developed in this way.

(I) Let \( \mathcal{M} \) be a collection of not necessarily complete varieties. Then strictly rational maps and proper birational maps are used as "rational maps" and "birational maps". For any variety \( V \) there exists a non-singular variety \( Y \) and a proper birational morphism \( \mu: Y \rightarrow V \). It may be natural to take \( Y \) as a non-singular model of \( V \). However, there exists a non-singular completion \( \overline{Y} \) with smooth boundary \( D \), which implies that \( Y = \overline{Y} - D \) and \( D \) has only simple normal crossings. It may be better to consider the triple \( (Y, \overline{Y}, D) \) as a nonsingular model of \( V \). In this case, "regular forms" of \( V \) are logarithmic forms on \( \overline{Y} \) with logarithmic poles along \( D \). Strictly speaking, let \( \Omega(\log D) \) denote the sheaf of germs of logarithmic 1-forms on \( Y \) with logarithmic poles along \( D \). For a poly-
nominal representation \( \varphi \) of \( \text{GL}(n) \), \( n \) being \( \dim V \), \( \Gamma(\mathcal{Y}, \Omega^n) \) is the space of logarithmic forms associated with \( \varphi \). Indeed, these vector spaces and their dimensions are invariants under any proper birational map. In particular, the Kodaira dimension of \( V \) is defined to be \( \kappa(K + D, \mathcal{Y}) \), where \( K \) is a canonical divisor on \( \mathcal{Y} \). This is denoted by \( \kappa(V) \). Replacing \( \kappa \) by \( \tilde{\kappa} \) and complete varieties by not necessarily complete ones, we have similar canonical fibering, called \textit{logarithmic canonical fibering}.

The easy addition theorem, the covering lemma and the existence of canonical fibering are formulated for the logarithmic Kodaira dimension and are easily proven. A logarithmic analogue of Conjecture \( C_n \), denoted by \( \tilde{C}_n \), seems more interesting. \( \tilde{C}_2 \) is completely solved and the structure theory of open surfaces, i.e., classification of surfaces in proper birational geometry, has been studied in detail ([8], [15]). Here the notion of numerical semipositivity and some variants of minimal models play indispensable roles (for example, the notion of an almost minimal model by Tsunoda). Even \( \tilde{C}_3 \) seems to be left unsolved. However, in the case where the base variety \( W \) is of general type, i.e., \( \kappa(W) = \dim W \), \( \tilde{C}_n \) has been recently verified by Maehara and Matsuda. For this, the weak-positivity of the direct image sheaf of the logarithmic relative canonical sheaf is a key fact (Viehweg [18], [19], Maehara).

(II). Affine varieties are special cases of non-complete varieties, hence they can be studied in birational geometry. But in some cases, normal varieties play roles of non-singular models.

(III). Considering pairs \((D, V)\) of normal varieties \( V \) and reduced divisors \( D \) on \( V \), one can develop the birational geometry of these pairs. For such a pair \((D, V)\), there exist a non-singular variety \( Y \) and a divisor \( B \) with simple normal crossings on \( Y \) and a birational map \( \mu: Y \to V \) such that the strict transform of \( B \) is \( D \). Then the logarithmic Kodaira dimension \( \tilde{\kappa}(Y-B) = \kappa(B+K(Y), Y) \) is a birational invariant; hence it is a birational invariant of \((D, V)\), which we denote by \( \kappa[D] \), ignoring \( V \). When \( V = \mathbb{P}^2 \), \( \kappa[D] = -\infty \) if and only if \( D \) is transformed into a line by a Cremona transformation. This is almost equivalent to Max Noether's theorem on the factorization of Cremona transformations. Further, plane curves are studied in the framework of birational geometry ([6], [7]).

References


