1. Introduction

This report will be on initial-boundary value problems for strictly hyperbolic equations. The particular form of the hyperbolic equation will not be important and without loss of generality one can consider a second order hyperbolic equation

$$A(x, D)u = 0$$  \hspace{1cm} (1)

in a cylindrical domain $\Omega = (-\infty, +\infty) \times G \subset \mathbb{R}^{n+1}$, where $x_0 \in (-\infty, +\infty)$ is a time variable and $(x_1, \ldots, x_n) \in G$ are space variables.

Solutions of (1) are subject to zero initial conditions

$$u = 0 \quad \text{for} \quad x_0 < 0, \quad x \in \Omega,$$  \hspace{1cm} (2)

and some boundary condition

$$B(x, D)u|_{\partial \Omega} = h(x'), \quad x' \in \partial \Omega,$$  \hspace{1cm} (3)

where $h = 0$ for $x_0 < 0$.

The problem is to find necessary and sufficient conditions on $B(x, D)$ such that the initial-boundary value problem (1), (2), (3) is well-posed. Note that all theorems that are formulated below will also apply to the case when (1) is a general hyperbolic equation or a hyperbolic system of equations of arbitrary order provided that all components of the characteristic cone are strictly convex (cf. [3]). Besides the Dirichlet and the Neumann conditions there are many boundary conditions that are of interest in mathematical physics, for example, (a) the impedance boundary condition

$$\frac{\partial u}{\partial n} - a(x) \frac{\partial u}{\partial x_0}|_{\partial \Omega} = h,$$  \hspace{1cm} (4)

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where $\partial/\partial v$ is the conormal derivative, (b) the boundary conditions in the linearized water wave theory, (c) the boundary conditions in elastodynamics for a solid with a free boundary. The last two examples describe interesting phenomena in wave propagation: supersonic boundary waves in the linearized water wave theory and Rayleigh’s waves (subsonic boundary waves) in elastodynamics.

2. Weak and strong Lopatinsky condition

In the theory of general boundary value problems for elliptic equations and initial-boundary value problems for parabolic equations, the following condition is necessary and sufficient for well-posedness:

Let $\tilde{a}$ be an arbitrary point of the boundary $\partial \Omega$. Freeze coefficients of $A(x, D)$ and $B(x, D)$ at the point $\tilde{a}$ and consider the constant coefficient boundary value problem for the principal parts of $A$ and $B$ in the half-space formed by the tangent plane to $\partial \Omega$ at the point $\tilde{a}$. This constant coefficient problem in the half-space can be solved explicitly using the Fourier transform. The algebraic condition of the well-posedness of the constant coefficient problem is called the Shapiro-Lopatinsky condition. If the Shapiro-Lopatinsky condition is satisfied for any $(x', \xi')$ in the cotangent bundle $T^*_0(\partial \Omega)$, where $x' \in \partial \Omega$ and $\xi' \neq 0$, then the boundary problem in $\Omega$ is well-posed. One can try the same criterion for the initial-boundary problem for hyperbolic equations. Since we know that $h$ and $u$ are zero for $x_0 < 0$, the Fourier transform with respect to $x_0$ will indeed be the Laplace transform, so that the variable dual to $x_0$ will be $\xi_0 + i\tau$ where $\tau > 0$. It can be shown (see [14]) that for the well-posedness of the initial-boundary value problem for hyperbolic equations, it is necessary that the Shapiro–Lopatinsky condition be satisfied for any $(x', \xi') \in T^*(\Omega)$ and any $\tau > 0$. This condition is called the weak Lopatinsky condition. It is necessary and sufficient for the well-posedness of IBVP (initial-boundary value problem) for hyperbolic equations with constant coefficients in the half-space. This result was established first by R. Hersch. As we shall see later, the weak Lopatinsky condition is not sufficient, in general, for the well-posedness of IBVP for hyperbolic equations. The reason is that there are boundary conditions that are sensitive to the local geometry of boundary (to convexity or concavity for instance). Therefore the tangent plane model is not good enough in these cases. Note that among such sensitive boundary conditions are the boundary conditions that produce boundary waves.
A general sufficient condition for the well-posedness of hyperbolic IBVP was found by Kreiss [15] and Sakamoto [21] in 1970. They proved that if the Shapiro–Lopatinsky condition is satisfied not only for all \((x', \xi') \in T^*_e(\Omega)\) and \(\tau > 0\) but also when \(\tau = 0\) then the hyperbolic IBVP is well-posed. Such a condition appeared first in the work of S. Agmon [1] and it is now called the Agmon–Kreiss–Sakamoto condition or the strong (or uniform) Lopatinsky condition. The strong Lopatinsky condition is satisfied in many problems of interest and it is independent of the shape of the boundary but there are important boundary conditions that do not belong to this class such as the Neumann boundary condition or any of the boundary conditions where boundary waves are present. An important class of boundary conditions for second order hyperbolic equations that implies an estimate of the solution in the energy norm was studied by S. Miyatake [19], L. Garding [9], and extended by R. Melrose and J. Sjostrand [18].

3. Microlocalization

For simplicity consider a boundary condition of the form

\[
\frac{\partial u}{\partial \nu} + \lambda(x', D')u|_{\partial \Omega} = h(x'), \tag{5}
\]

where \(\partial/\partial \nu\) is the conormal derivative and \(\lambda(x', D')\) is a first order differential or pseudodifferential operator in tangential variables. Although the operator \(\lambda\) is not pseudodifferential in physical applications for second order hyperbolic equations, pseudodifferential \(\lambda\) arises when one considers a hyperbolic equation of higher order or a system of hyperbolic equations. Then after a microlocalization the problem is reduced to a pseudodifferential equation \(A(x, D)\) of the second order that is differential in the normal variable and the boundary operator has the form (5) with a pseudodifferential \(\lambda(x', D')\). There is a natural partition of \(T^*_e(\partial \Omega)\) into three regions: (1) the elliptic region where the principal symbol of \(A(x, D)\) has no real zeros with respect to a variable dual to the normal, (2) the hyperbolic region where there are two distinct real zeros, and (3) the diffraction region where there is one double real zero.

In the elliptic and hyperbolic region the investigation of the IBVP can be reduced to a study of a pseudodifferential equation on the boundary. This reduction was done first by P. D. Lax and L. Nirenberg (see [20]).
4. Case of a strictly concave boundary

The most difficult part of the problem is the study of the neighborhood of the diffraction region. We shall consider this problem under the additional restriction that the boundary $\partial \Omega$ is either strictly convex or strictly concave with respect to the bicharacteristics of the hyperbolic operator $A(x, D)$ that are tangent to $\partial \Omega$. Indeed one needs the concavity or convexity conditions only on the intersection of the set where the strong Lopatinsky condition does not hold with the diffraction region because in the region of $T^*_0(\partial \Omega)$ where the strong Lopatinsky condition holds one can use the Kreiss method.

Important examples of the IBVP with a convex or a concave boundary are the initial-boundary value problem for the wave equation in the interior or in the exterior of a strictly convex bounded domain in $\mathbb{R}^n$.

The following theorem describes conditions of the well-posedness of IBVP in the case of concave boundary.

**Theorem 1** (see [6]). Assume, for simplicity, that the boundary condition has form (5) and it is of principal type with respect to $x_0$ in the elliptic and hyperbolic region. Then the weak Lopatinsky condition is necessary and sufficient for the well-posedness of IBVP with a concave boundary.

5. Hypoelliptic boundary conditions

There is an important class of boundary conditions (5) which are called **hypoelliptic** (see [18]).

The hypoellipticity of the boundary condition means that for any distribution solution $u$ of (1), (2), (3) the following inclusion holds:

$$WF(u|_{\partial \Omega}) \subset WF(h),$$

where $u|_{\partial \Omega}$ is the restriction of $u$ on $\partial \Omega$ and $WF(u|_{\partial \Omega}) \subset T^*_0(\partial \Omega)$ and $WF(h) \subset T^*_0(\partial \Omega)$ are the wave front sets.

For example, the Neumann problem is hypoelliptic in the case of a concave boundary (see [16] and [22]). R. Melrose and J. Sjostrand (see [18], Part II) found a sufficient condition for the hypoellipticity of the IBVP and they formulated the following general conjecture that was proven in [8]:

**Theorem 2.** The boundary condition (5) in the case of concave boundary is hypoelliptic if the strong Lopatinsky condition holds in the elliptic and hyperbolic regions and for any point $(\tilde{x}, \tilde{\xi})$ in the diffraction region where
the strong Lopatinsky condition does not hold there is a neighborhood $U_0$ in the diffraction region such that

$$-\frac{\pi}{2} + \varepsilon_0 \leq \arg \lambda_1 \leq \pi - \varepsilon_0, \quad \varepsilon_0 > 0, \quad (7)$$

for any $(x', \xi') \in U_0$, where $\lambda_1$ is the restriction of the principal part of $\lambda(x', \xi)$ to the diffraction region.

6. Case of a convex boundary

The case of a convex boundary is more complicated than the case of a concave boundary because of multiple reflections of waves. The following general result was proven in [8]:

**Theorem 3.** Assume that the boundary is convex and the weak Lopatinsky condition is satisfied. Let, for simplicity, the boundary operator $(5)$ be of principal type with respect to $x_0$ in the elliptic and hyperbolic regions. Assume that for any point $(\tilde{x}, \tilde{\xi})$ in the diffraction region where the strong Lopatinsky condition does not hold there is a neighborhood $U_0$ in the diffraction region such that

$$-\Re \lambda_1 \leq C(\Im \lambda_1)^2 |\xi'|^{-1} \ln \frac{1}{|\Re \lambda_1| |\xi'|^{-1}}, \quad (8)$$

for all $(x', \xi') \in U_0$, where $\lambda_1$ is the same as in Theorem 2. Then the IBVP $(1), (2), (5)$ is well-posed.

It was shown in [7] that the conditions of Theorem 3 are necessary and sufficient for the well-posedness of a model problem with a convex boundary. For a general hyperbolic equation the situation is more complicated. Consider, for simplicity, the case when $\lambda_1$ is a real-valued symbol. Then the condition $(8)$ simply means that $\lambda_1 \geq 0$ in $U_0$.

**Theorem 4** (see [8]). Let $\lambda_1$ be real and assume that the condition $(8)$ is not satisfied at the point $(\tilde{x}, \tilde{\xi})$ in the diffraction region. Let the Poisson bracket

$$\{\lambda_1, \mu\} = 0 \quad \text{at the point } (\tilde{x}, \tilde{\xi}), \quad (9)$$

where $\mu = 0$ is the equation of the diffraction region. If the boundary is strictly convex at the point $(\tilde{x}, \tilde{\xi})$ then the IBVP is ill-posed.
Therefore the conditions for well-posedness of IBVP with a convex boundary are very restrictive when the strong Lopatinsky condition fails. Nevertheless it was shown in [7] for a model problem with a convex boundary that if the condition (8) is not satisfied but the Poisson bracket (9) is not equal to zero then the IBVP is still well-posed.

7. Examples

Let (1) be the wave equation and the boundary condition has the form (4) (the impedance boundary condition). Then the weak Lopatinsky condition has the form

\[ a(x') > -1 \quad \text{on } \partial \Omega. \tag{10} \]

It follows from Theorem 1 that for the wave equation in the exterior of a strictly convex domain with the boundary condition (4) the condition (10) is necessary and sufficient for the well-posedness of IBVP (see also M. Ikawa [10]). Consider the same problem in the interior of a convex domain. If \( a(\tilde{x}) = 0 \) and \( \frac{\partial}{\partial \omega_0} a(\tilde{x}) = 0 \) for some point \( \tilde{x} \in \partial \Omega \) and if \( a(x') \) is not a nonnegative function in a neighborhood of \( \tilde{x} \) (i.e. the condition (8) is not satisfied) then Theorem 4 implies that IBVP is ill-posed.

As a second example, consider the case of the oblique derivative boundary condition

\[ \frac{\partial u}{\partial \nu} + \tau u |_{\partial \Omega} = h, \tag{11} \]

where \( \partial / \partial \nu \) is the interior conormal derivative and \( \tau \) is a tangential vector field on \( \partial \Omega \).

The exterior problem with the boundary condition (11) is always well-posed. This fact was proved by M. Ikawa (see [10]). But the interior problem is ill-posed when \( \tau \) is not identically zero and has a degenerate critical point. Analogous results hold for the transmission problem

\[ A_1(x, D) u_1 = 0 \quad \text{in } \Omega, \tag{12} \]
\[ A_2(x, D) u_2 = 0 \quad \text{in } \Omega \Omega, \]

with the transmission conditions

\[ u_1 |_{\partial \Omega} = u_2 |_{\partial \Omega}, \tag{13} \]

and

\[ \frac{\partial u_1}{\partial \nu_1} + \tau_1 u_1 |_{\partial \Omega} = - \frac{\partial u_2}{\partial \nu_2} + \tau_2 u_2 |_{\partial \Omega}, \tag{14} \]
where $\partial \Omega$ is the complement to $\Omega$ in $\mathbb{R}^{n+1}$, $\partial / \partial \nu_1$ and $\partial / \partial \nu_2$ are the interior conormal derivatives with respect to $A_1$ and $A_2$, $\tau_1$ and $\tau_2$ are tangential vector fields to $\partial \Omega$ and we assume that $\partial \Omega$ is strictly convex with respect to the tangential bicharacteristics of $A_1$ and is strictly concave with respect to the tangential bicharacteristics of $A_2$. Then the transmission problem (12), (13), (14) with zero initial conditions is ill-posed if $\tau_1 - \tau_2$ is not equal to zero identically and there exists a degenerate critical point of $\tau_1 - \tau_2$.

8. Propagation of singularities

There is a close relation between the well-posedness of IBVP and the propagation of singularities. R. Melrose [16] and M. Taylor [22], first completely described the singularities of IBVP with concave boundaries for the cases of the strong Lopatinsky boundary conditions and the Neumann boundary condition.

The case of IBVP with convex boundaries for the Dirichlet and the Neumann boundary conditions was done independently by K. G. Andersson and R. Melrose [2], G. Eskin [4] and V. Ia. Ivrii [12]. It was shown in these works that the singularities propagate along broken bicharacteristics that are undergoing multiple reflections on the boundary and along the gliding rays. Further important progress was made by R. Melrose and J. Sjöstrand in [18] where the propagation of singularities for general domains were studied without restriction on convexity or concavity of the boundary.

Under the restriction that the boundary is concave, the propagation of singularities for an arbitrary boundary operator with a real-valued symbol satisfying the weak Lopatinsky condition was studied in [6]. For such a general boundary condition the boundary waves may appear but they do not represent a threat to the well-posedness of IBVP.

In the case of convex boundaries there is no hypoelliptic boundary condition in the sense of the definition (6) because there is always a propagation of singularities along the gliding rays. When the conditions of Theorem 3 are satisfied and the strong Lopatinsky condition holds in the elliptic and the hyperbolic regions, then the only singularities of $u|_{\partial \Omega}$ are contained in the union of all gliding rays and broken bicharacteristics that start at the points of $WF(h)$.

If the condition (8) is not satisfied then the picture of the propagation of singularities is more complicated. The singularities of $u|_{\partial \Omega}$ come from boundary waves propagating along the boundary, and waves propaga-
ting in $Q$ and undergoing multiple reflections. In general (when the condition (9) holds) that leads to singularities so strong that the solution of IBVP ceases to be a distribution. And this is a reason for the ill-posedness of IBVP under the conditions of Theorem 4. Indeed the solution becomes an ultradistribution, i.e. a functional over the space of $C^\infty$ functions that belong to a certain Gevrey class.

References


