Long Time Behaviour of Solutions to Nonlinear Wave Equations

Most basic equations of both physics and geometry have the form of nonlinear, second order, autonomous systems

\[ G(u, u', u'') = 0, \]

where \( u = u(x^1, x^2, \ldots, x^{n+1}) \), and \( u', u'' \) denote all the first and second partial derivatives of \( u \). For simplicity we will assume here that both \( u \) and \( G \) are scalars and denote by \( u_a, u_{ab} \), the partial derivatives \( \partial_a u \) and respectively \( \partial^2_{ab} u; a, b = 1, 2, \ldots, n+1 \). Let \( u^0(x) \) be a given solution of (1). Our equation is said to be elliptic or hyperbolic at \( u^0(x) \) according to whether the \((n+1) \times (n+1)\) matrix, whose entries are \( G_{ab} = \frac{\partial G}{\partial u_{ab}} (u^0, u'^0, u''^0) \), is nondegenerate and has signature \((1, \ldots, 1, 1)\) or \((1, \ldots, 1, -1)\). Nonlinear elliptic equations and systems have received a lot of attention in the past forty or fifty years and in this period a lot of progress was made and powerful methods were developed. By comparison, the field of nonlinear hyperbolic equations is wide open. In what follows I will try to point out some recent developments concerning long-time behavior of smooth solutions to a large class of such equations.

Let us assume that \( G(0, 0, 0) = 0 \) and that (1) is hyperbolic around the trivial solution \( u^0 = 0 \). Typically, the operator obtained by linearizing (1) around \( u^0 = 0 \) contains only second derivatives. Without further loss of generality we may assume it to be the wave operator \( \partial^2_1 + \cdots + \partial^2_n - \partial^2_t = -\Box \), where we have ascribed to \( x_{n+1} \) the role of the time variable \( t \).

The equation (1) takes the form (1')

\[ \Box u = F(u, u', u'') \]

with \( F \) a smooth function of \( (u, u', u'') \), independent of \( u_{tt} \), vanishing together with all its first derivatives at \((0, 0, 0)\).
Associate to (1') the pure initial value problem
\[ u(x, 0) = ef(x), \quad u_t(x, 0) = sg(x) \] (1'a)
with \( f, g, C^\infty \)-functions, decaying sufficiently fast at infinity (for simplicity, say \( f, g \in C^\infty_0(\mathbb{R}^n) \)) and \( \varepsilon \) is a parameter which measures the amplitude of the data. Given \( f, g \) and \( F \) we define the life span \( T_\varepsilon = T_\varepsilon(\varepsilon) \) as the supremum over all \( T \geq 0 \) such that a \( C^\infty \)-solution of (1'), (1'a) exists for all \( x \in \mathbb{R}^n, 0 \leq t < T \). The fundamental local existence theorem ([3], [4], [14], [24], [27]), asserts that, if \( \varepsilon \) is sufficiently small, so that the initial data lies in a neighborhood of hyperbolicity of the zero solution, then \( T_\varepsilon(\varepsilon) > 0 \). Moreover, a simple analysis of the proof shows that \( T_\varepsilon(\varepsilon) \geq \frac{A}{\varepsilon} \)
where \( A \) is some small constant, depending only on a finite number of derivatives of \( F, f, g \). This lower bound for \( T_\varepsilon \) is in general sharp if the number of space dimensions \( n \) is equal to one. Indeed let our variables in (1') be \( x \) and \( t \) and \( F = \sigma(u_x)u_{xx} \) with \( \sigma \) a smooth function, \( \sigma(0) = 0 \). An old result of P. Lax [22], extended to systems of wave equations by F. John [10], shows that, under the assumption of "genuine nonlinearity", \( \sigma'(0) \neq 0 \), all solutions to the corresponding initial value problem (1'a) blow up by the time \( O\left(\frac{1}{\varepsilon}\right) \). Recently, in [20], it was proved that \( T_\varepsilon < \infty \) even if the genuine nonlinearity condition is violated. More precisely, assume that \( \sigma'(0) = \ldots = \sigma^{(P)}(0) = 0, \sigma^{(P+1)}(0) \neq 0 \) then the corresponding solutions blow up by the time \( T = O(1/\varepsilon^{P+1}) \). In both situations the blow-up occurs in the second derivatives of \( u \) i.e. \( u_{xx} \) becomes infinite while \( u_t, u_x \) remain bounded. Such blow-ups are typical of shock formations and are observable phenomena of physical reality. If the original equation, or system can be written in conservation form i.e., in our case,
\[ F(u, u', u'') = \sum_{a=1}^{n+1} \partial_a f^a(u, u'), \]
one can try to extend the solutions past these breakdown points by introducing the concept of weak solutions. This was successfully accomplished for very general first order systems of conservation laws, in one space dimension, by the fundamental work of Oleinik, P. Lax and J. Glimm (see [24] for a bibliography). In this lecture I will restrict myself, however, to classical, i.e. \( C^\infty \)-solutions.

Surprisingly, the situation looks better in higher dimensions. In 1976 F. John [9] proved that, under the assumption \( F = F(u', u'') \), and \( n \geq 3 \), \( T_\varepsilon(\varepsilon) \) can be significantly improved and, in 1980, S. Klainerman [15] was able to push \( T_\varepsilon(\varepsilon) \) to infinity, and thus obtain global solutions, provided that \( n \geq 6 \). More generally, see ([17], [21], [28]),
Theorem 1. Assume that \( F = F(u', u'') = O(|u'| + |u''|)^{p+1} \) for small \( u', u'' \) and that \( \frac{1}{p} \left( 1 + \frac{1}{p} \right) < \frac{n-1}{2} \), then there exists an \( \varepsilon_0 \) sufficiently small such that for all \( \varepsilon \leq \varepsilon_0 \), (1'), (1'a) has a unique smooth solution for all \( x \in \mathbb{R}^n, t \geq 0 \).

The reason for this improved behavior of solutions of (1') in higher dimensions was beautifully illustrated by F. John [9] with the following quotation from Shakespeare, Henry VI:

"Glory is like a circle in the water,
Which never ceaseth to enlarge itself,
Till by broad spreading it disperse to naught".

Indeed, the higher the dimension the more room for waves to disperse and thus decay. Accordingly, the key to [9], [15], [17], [21] and [28] is to use decay estimates for solutions to the classical wave equation, \( u = 0 \) (see [25], [26], [31]), and to combine them with energy estimates for higher derivatives of solutions to the original, nonlinear equations.

The dimension \( n = 3 \), which nature gives preference, is not only the most important but also the most challenging. In [7] F. John exhibited an example for which \( T_* < \infty \). More precisely consider \( F = u_t \cdot u_{tt} \) and the corresponding equation (1') in three space dimensions. Imposing only one mild restriction on the data, \( \int g(x) \, dx > 0 \), F. John showed that there are no \( C^2 \)-solutions defined for all \( x \in \mathbb{R}^3, t \geq 0 \). However, for sufficiently small \( \varepsilon \), the solutions remain smooth for an extremely long-time before a breakdown occurs. We have in fact the following very general.

Theorem 2 (F. John, S. Klainerman). Assume that \( F \) verifies one of the following hypothesis:

(H_1) \( F \) does not depend explicitly on \( u \) i.e., \( F = F(u', u'') \)

(H_2) \( F \) can be written in conservation form,

\[
F(u, u', u'') = \sum_{a=1}^{4} \partial_a f^a(u, u'),
\]

(H'_2) \( F(u, u', u'') = \sum_{a=1}^{4} \partial_a f^a(u, u') + O(|u| + |u'| + |u''|)^3
\]

for small \( u, u', u'' \)

\[1\] The author was recently able to improve this result [32]. The sharp condition which assures global existence is \( p > 2/(n-1) \).
Then, there exist three small, positive constants $s_0$, $A$, depending only on a finite number of derivatives of $F$, $f$, $g$, such that for every $0 < s < s_0$, $T_s(e) \geq \exp(A/e)$.

Previously a weaker, polynomially long time existence result, was proved by F. John in [12] using an asymptotic expansion in powers of $s$, for $u$ (see also [9]). The exponential long-time existence result was first proved, for spherically symmetric solutions (in the semilinear case $F = F(u')$, by F. John [7] and T. Sideris [29], and for $F = F(u', u'')$ by S. Klainerman [18]).

The result of Theorem 2 is in general sharp. Indeed, F. John [11] proved recently that this is the case in the context of his previous example, $F = u_1 u_2$. There is, however, quite a rich class of nonlinearities $F$ for which global existence holds. The following can be regarded as a generalization of Theorem 1 in dimension $n = 3$.

**Theorem 3.** Assume that $F$ verifies the following "Null"-condition

\[(N) \quad \sum_{a,b,c=1}^{4} (e_{ab}e_{cd}e_{ac} - e_{ad}e_{bc}e_{bc}) = O(|u| + |u'| + |u''|)^3 \]

for every sufficiently small $u$, $u'$, $u''$ and any fixed null space-time vector $(X^1, X^2, X^3, X^4)$ i.e., $(X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 = 0$. Then, if $e$ is sufficiently small, a global smooth solution of (1'), (1'a) exists.

To illustrate the content of Theorem 2 note that either of the following examples verifies the condition (N)

**Example 1** $F = u_a u_{bc} - u_b u_{ac}$, for any three indices $a$, $b$, $c = 1, 2, 3, 4$.

**Example 2** $F = \partial_a(u_i^2 - u_1^2 - u_2^2 - u_3^2)$, for any index $a = 1, 2, 3, 4$.

The proof of both Theorems 2 and 3 depends on some recent [19] weighted $L^\infty$ and $L^1$ estimates for solutions to the classical, inhomogeneous, wave equation in dimension $n = 3$. They were first used, in the spherical symmetric case, in [18] and then extended to the general case by introducing the angular momentum operators $\Omega_1 = x_3 \partial_2 - x_2 \partial_3$, $\Omega_2 = x_1 \partial_3 - x_3 \partial_1$, $\Omega_3 = x_2 \partial_1 - x_1 \partial_2$. Their key property is that they commute with the wave operator $\square$ and thus can be treated as the usual partial derivat-
In particular, this allows us to extend the energy estimates used in [9], [15], [17], [21] and [28], to any combination of the derivatives $\partial_1, \partial_2, \partial_3, \partial_4, \partial_5, \partial_6, \partial_7, \partial_8$. The operators $\Omega_1, \Omega_2, \Omega_3$ are closely connected to the "radiation operators" $L_1, L_2, L_3$ which played a fundamental role in [12].

A different, and very interesting proof of Theorem 3, based on some conformal mapping methods, was given by D. Christodoulou [1]. (See also his previous joint work with Y. Choquet-Bruhat [2].)

Both Theorems 2 and 3 have straightforward extensions to systems, in particular to those of the type arising in Nonlinear Elasticity and General Relativity. There are important problems, like that of stability of the Minkowski Space as a solution of the Einstein equations in vacuum, for which we hope that Theorems 2 and 3 could be relevant. In the scalar case we believe that the picture provided by these theorems, together with the nonexistence results of F. John [7], [11] can be completed. In other words, we conjecture that if one of the hypothesis (H1), (H2), (H2') holds and (N) fails, then the lower bound on $T_* (\varepsilon)$ given by Theorem 2 is sharp, i.e. singularities must develop by that time, for any choice of $f$ or $g$ and $\varepsilon$ small. An important open question is to describe the type of blow-up which occurs in that case. If $F$ is quasilinear and verifies $H1$, we expect that, as for $n = 1$, the breakdown occurs when the second derivatives of $u$ become infinite while the first derivatives remain bounded.

The recent work of F. John [28] points in this direction, but completely satisfactory results are still missing. Another open question is to derive results similar to Theorems 2 and 3 for the dimensions $n = 2$ and $n = 4$. We suspect that the corresponding, optimal lower bound for $T_*$ for $n = 2$ must be $O \left( \frac{1}{\varepsilon^2} \right)$ while for $n = 4$ one should be able to prove global existence.²

In this respect we hope to find decay estimates similar to those of [19] for $n \neq 3$. The same type of questions can be asked for equations (1') where the wave operator $\Box$ is replaced by the Klein–Gordon operator, $\Box + m^2$ or the Schrödinger operator $-i \partial_t + A$. General results of the type of Theorem 1 were derived in [17], [21], [28], and for nonlinearities depending only on $u$ in [30], (see also the reference there). The methods used to derive Theorems 2 and 3 might be used to substantially improve these results.

In the end, I like to apologize for not mentioning the work of many

² See footnote, p. 1211.
other authors. In particular I have left out a lot of interesting results concerning semilinear equations i.e. $F = F(u)$ in (1'). For an up to date bibliography concerning such results I refer to the recent papers of R. Glassey [5], [6].

References

[8] John F., Blow-up of Radial Solutions of $\Box u = -\frac{\partial F(u)}{\partial u}$, in preparation.


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Added in proof:
