Introduction

In many applications (engineering, management, economy) one is led to control problems for stochastic systems: more precisely the state of the system is assumed to be described by the solution of stochastic differential equations and the control enters the coefficients of the equation. Using the dynamic programming principle R. Bellman [6] explained why, at least heuristically, the optimal cost function (or value function) should satisfy a certain partial differential equation called the Hamilton–Jacobi–Bellman equation (HJB in short), which is of the following form

\[ \sup_{\alpha \in \mathcal{A}} \{ A_{\alpha} u - f_{\alpha} \} = 0 \text{ in } \varnothing \]  

(with appropriate boundary conditions) where \( A_{\alpha} \) is a family of second-order, elliptic, possibly degenerate operators, parametrized by \( \alpha \) lying in a given set \( \mathcal{A} \) (of the control values); and where \( f_{\alpha} \) is a family of given functions. Here and below \( \varnothing \) is a given domain in \( \mathbb{R}^N \) and \( u \) is a scalar function.

The HJB equations are second-order, degenerate elliptic, fully nonlinear equations of the following form

\[ H(x, u, Du, D^2 u) = 0 \text{ in } \varnothing \]

with the main restriction that \( H \) is convex in \((Du, D^2 u)\).

As special cases the HJB equations (1) include

(i) the first-order Hamilton–Jacobi equations (HJ in short)

\[ H(x, u, Du) = 0 \text{ in } \varnothing. \]  

[1403]
Strictly speaking, (1) contains (2) when \( H \) is convex in \((t, p)\), but as it will be made clear below, our methods enable us to treat the general HJ equation (2), i.e., the case of a general Hamiltonian \( H \).

(ii) the Monge–Ampère equations

\[
\det(D^2 u) = H(x, u, \nabla u) \quad \text{in } \varnothing, \quad u \text{ convex in } \overline{\varnothing}. 
\]  

(3)

Again strictly speaking, (3) is a special case of (1) only if \( H(x, t, p) \) is convex in \((t, p)\) — and if this is the case, the fact that (1) contains (3) is indicated in Section IV.1. But just as above, the methods we give below enable us to treat the general Monge–Ampère equations (3).

We present here various existence and uniqueness results for equations (1)–(2)–(3), and one consequence of the results presented below is a complete justification of the derivation of (1) in the theory of optimal stochastic control. The tools and methods that we used or introduced for this study are of three kinds:

(i) the notion of viscosity solutions of (1)–(2): this notion, introduced by M.G. Crandall and the author, makes possible, in particular, a complete treatment of the HJ equations (2);

(ii) probabilistic methods: many of them being inspired by N.V. Krylov’s work;

(iii) new partial differential equation arguments involving approximation methods and a priori estimates.

The plan is as follows

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Bibliography
I. Viscosity solutions

1.1 Definition of viscosity solutions for Hamilton-Jacobi equations. We recall briefly below the notion of viscosity solutions of HJ equations introduced by M.G. Crandall and the author [14]. This notion enables us to settle the question arising from the following remarks: (i) in general, there does not exist global $C^1$ solutions, (ii) if $W^{1,\infty}$ solutions of (2) can easily be built by the vanishing viscosity method (see W. H. Fleming [22]), then in general, there may exist many $W^{1,\infty}$ solutions of (2) with prescribed boundary conditions; and moreover, the Lipschitz solutions are unstable, see [14] for more details. The notion of viscosity solutions enables us to select the 'good' solution for which existence and uniqueness results hold. In addition one has stability results and viscosity solutions are exactly the solutions built by the vanishing viscosity method.

Let $\mathcal{O}$ be an open set in $\mathbb{R}^N$ and let $H$ be a continuous function on $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^N$. We first recall the notion of sub and superdifferential of a continuous function $u$ at a point $x_0 \in \mathcal{O}$: the superdifferential $D^+_u(x)$ is the closed convex set, possibly empty, defined by

$$D^+_u(x) = \big\{ \xi \in \mathbb{R}^N \big| \lim_{y \to x_0, y \in \mathcal{O}} \sup_{\psi \in \partial u(x_0)} \{ \psi(y) - \psi(x_0) - \langle \xi, y-x_0 \rangle \} |y-x|^2 \leq 0 \big\};$$

the subdifferential $D^-_u(x)$ being defined in a similar way or by

$$D^-_u(x) = -D^+_u(-u)(x).$$

Definition 1.1. $u \in C(\mathcal{O})$ is said to be a viscosity solution of the HJ equation (2) if for all $x \in \mathcal{O}$ we have

$$\forall \xi \in D^+_u(x), \quad H(x, u(x), \xi) \leq 0, \quad (4)$$

$$\forall \xi \in D^-_u(x), \quad H(x, u(x), \xi) \geq 0. \quad (5)$$

Remark 1.1. Of course, if $\varphi \in C(\mathcal{O})$ is differentiable at $x_0 \in \mathcal{O}$ then $D^+_u(x_0) = D^-_u(x_0) = \{ \varphi'(x_0) \}$; and conversely, if $D^+_u(x_0) \cap D^-_u(x_0) \neq \emptyset$ then $\varphi$ is differentiable at $x_0$. In particular, any classical (i.e., $C^1$) solution of (2) is a viscosity solution of (2), and any viscosity solution $u$ of (2) satisfies equation (2) at all points of differentiability.

This notion, introduced by M.G. Crandall and P.L. Lions [14] has many equivalent formulations; the most 'convenient' one being given in the following
**Proposition I.1.** Let \( u \in C(\theta) \); \( u \) is a viscosity solution of (2) if and only if, for all \( \varphi \in C^1(\theta) \),

at each local maximum point \( x_0 \) of \( u - \varphi \) we have

\[
H(x_0, u(x_0), D\varphi(x_0)) \leq 0, 
\]

(4′)

at each local minimum point \( x_0 \) of \( u - \varphi \) we have

\[
H(x_0, u(x_0), D\varphi(x_0)) \geq 0. \quad \blacksquare 
\]

(5′)

**Remark I.2.** It is possible to replace in the above statement local by global (resp. global strict, resp. local strict) and \( \varphi \in C^1 \) by \( \varphi \in C^2 \) (resp. \( \varphi \in C^{2\theta} \)). For more details, we refer to [14] and to M. G. Crandall, L. C. Evans, and P. L. Lions [13].

One of the striking features of viscosity solutions is their stability with respect to uniform convergence (on compact sets): if \( u_n \in C(\theta) \) is a viscosity solution of (2) where \( H \) is replaced by \( H_n \), and if \( u_n, H_n \) converge uniformly on compact sets to \( u, H \), then \( u \) is a viscosity solution of (2).

Similar results are obtained for the limit functions obtained via the vanishing viscosity method: if \( u_\varepsilon \in C^2(\theta) \) solves

\[
-\varepsilon \Delta u_\varepsilon + H_\varepsilon(x, u_\varepsilon, Du_\varepsilon) = 0 \quad \text{in } \theta
\]

and if \( u_\varepsilon, H_\varepsilon \) converge uniformly on compact sets to \( u, H \) as \( \varepsilon \) goes to 0, then \( u \) is a viscosity solution of (2).

This simple remark enables us to obtain very general existence results of viscosity solutions for the HJ equation (2) (with prescribed boundary conditions on \( \partial \theta \)): using the vanishing viscosity method and the properties stated above, this amounts to the obtention of a priori \( W^{1,\infty} \) estimates uniform in \( \varepsilon \). Existence results are treated in P. L. Lions [33], [35].

### I.2. Some of the main results on viscosity solutions of HJ equations

We now present a uniqueness result taken from M. G. Crandall and P. L. Lions [14]. We use the following assumptions

\[
\exists \gamma = \gamma(R) > 0, \quad \forall (x, p) \in \theta \times \mathbb{R}^N, \quad \forall |t|, |s| \leq R,
\]

\[
(H(x, t, p) - H(x, s, p))(t-s) \geq \gamma(t-s)^2, \quad (6)
\]

\[
\lim_{\varepsilon \to 0} \left[ \sup_{|x-y| \leq \varepsilon} |H(x, t, p) - H(y, t, p)| \right] = 0 \quad (7)
\]

for all \( R < \infty \).
THEOREM I.1. Assume that \( \mathcal{O} \) is bounded and that (6) holds. Let \( u, v \in C(\bar{\mathcal{O}}) \) be two viscosity solutions of (2). We assume in addition either that (7) holds or that \( u, v \in W^{1,\infty}(\mathcal{O}) \). Then the following inequality holds:
\[
\sup_{\mathcal{O}} (u - v)^+ \leq \sup_{\partial \mathcal{O}} (u - v)^+.
\]

Of course, (8) implies uniqueness results for viscosity solutions of (2) with prescribed boundary conditions on \( \partial \mathcal{O} \): indeed, if \( u = v \) on \( \partial \mathcal{O} \), then in view of (8), \( u = v \) in \( \bar{\mathcal{O}} \).

Remark I.3. This result is shown in [14], [33] to be essentially optimal; variants concerning unbounded domains such as \( \mathbb{R}^N \) or time-dependent problems are given in [14].

I.3. Remarks on the viscosity solutions for second-order equations. We now consider fully nonlinear second-order elliptic equations such as
\[
H(x, u, Du, D^2 u) = 0 \quad \text{in} \ \mathcal{O};
\]
where \( H \in C(\mathcal{O} \times \mathbb{R} \times \mathbb{R}^N) \) satisfies the following ellipticity condition:
\[
H(x, t, p, \eta_1) \leq H(x, t, p, \eta_2) \quad \text{if} \quad \eta_1 \geq \eta_2, \ \forall (x, t, p) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}^N.
\]

To define viscosity solutions of (9), we must first define, for all \( (u, x_0) \in C(\mathcal{O}) \times \mathcal{O} \), the superdifferential of order 2, \( D^+_2 u(x_0) \). It is the closed convex set, possibly empty, defined as follows:
\[
D^+_2 u(x_0) = \{(\xi, \eta) \in \mathbb{R}^N \times S^N | \lim_{\nu \rightarrow x_0, v \in \mathcal{O}} \sup \{u(y) - u(x_0) - (\xi, y - x_0) - \frac{1}{2} \eta (y - x_0, y - x_0) |y - x_0|^2 \leq 0\},
\]
the subdifferential of order 2 \( D^-_2 u(x_0) \) being defined in a similar way or by
\[
D^-_2 u(x_0) = -D^+_2 (-u)(x_0).
\]

DEFINITION I.2. \( u \in C(\mathcal{O}) \) is a viscosity solution of (9) if for all \( x \in \mathcal{O} \) we have
\[
\forall (\xi, \eta) \in D^+_2 u(x), \quad H(x, u(x), \xi, \eta) \leq 0,
\]
\[
\forall (\xi, \eta) \in D^-_2 u(x), \quad H(x, u(x), \xi, \eta) \geq 0. \quad \blacksquare
\]

\(^1 S^N \) denotes the space of \( N \times N \) symmetric matrices.
It can easily be checked that Remarks 1.1–2 and the equivalent formulation of Proposition I.1 can be extended to this case. Of course, if $H$ does not depend on $D^2u$, we recover the preceding notion. The stability results also hold if we consider sequences of $H, u$ converging uniformly on compact sets. For more details on these questions we refer to P.L. Lions [34], [38].

The main question in this case concerns uniqueness results: except for easy results (if $N = 1, 2, \ldots$), the only known case is when (9) reduces to the HJB equation (see Section II. 3 below). As regards existence results, general ones may be obtained by appropriate approximation methods.

I.4. Further results. In the bibliography, various references are given concerning the notion of viscosity solutions for HJ equations and its applications to existence results, numerical approximation, optimal deterministic control problems, asymptotic problems, nonlinear semigroup theory, accretive operators. In P.L. Lions [39], the relations between viscosity solutions of (9) and $W^{2,p}$-solutions of (9) (satisfying (9) a.e.) are investigated.

II. Optimal stochastic control problems

II.1. Presentation of the problem. We define an admissible controlled system as a collection consisting of (i) a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with the usual properties, (ii) an $\mathcal{F}_t$-Brownian motion $\mathbf{B}_t$, (iii) a progressively measurable process $\alpha_t$ with compact values in a given separable metric space $\mathcal{M}$; here $\alpha_t$ is the control process. The state of the system is given by the solution of the following stochastic differential equation

$$dX_t = \sigma(X_t, \alpha_t)dB_t + b(X_t, \alpha_t)dt, \quad x_0 = x \in \mathcal{O},$$

(13)

where, for simplicity, $\mathcal{O}$ is a bounded smooth domain in $\mathbb{R}^N$ and $\sigma(x, \alpha) = (\sigma_i(x, \alpha))_{1 \leq i \leq N}, b(x, \alpha) = (b_i(x, \alpha))_{1 \leq i \leq N}$ are coefficients satisfying conditions detailed below and $m$ is a given integer. We now define the cost functions and the value function of the problem

$$J(x, \mathcal{P}) = E \left\{ \int_0^t f(X_s, \alpha_s) \exp \left( - \int_0^s \sigma(X_u, \alpha_u) \, du \right) \, ds \right\},$$

$$+ \varphi(X_T) \exp \left( - \int_0^T \sigma(X_u, \alpha_u) \, du \right),$$

(14)

$$u(x) = \inf \{ J(x, \mathcal{P}) | \mathcal{P} \text{ admissible system} \},$$

(15)
where \( f(x, \alpha), c(x, \alpha) \) are given real-valued functions and \( \tau \) is the first exit time from \( \partial \) of \( X_t \); \( \tau = \inf\{t \geq 0, X_t \notin \partial \} \). To simplify the presentation we will always assume that

\[
\begin{aligned}
\psi(\cdot, \alpha) &\in W^{2,\infty}(\mathbb{R}^N), \quad \varphi \in W^{3,\infty}(\mathbb{R}^N), \quad \sup_{\alpha, \varphi} \|\psi(\cdot, \alpha)\|_{W^{2,\infty}} < \infty, \\
\psi(x, \cdot) &\in C(\mathcal{A}) \text{ for all } x \in \mathbb{R}^N, \text{ for } \varphi = \sigma_i, b_i, c, f; \\
\lambda &\equiv \inf\{c(x, \alpha)/x \in \mathbb{R}^N, \alpha \in \mathcal{A}\} > 0.
\end{aligned}
\]

(16)

(17)

We now want to study the value function \( u \) and to show that, in a suitable sense, \( u \) satisfies and is characterized by (1):

\[
\sup_{\alpha \in \mathcal{A}} [A_{\alpha} u - f_{\alpha}] = 0 \text{ in } \mathcal{O},
\]

(1)

where \( f_{\alpha}(x) = f(x, \alpha), A_{\alpha} = -a_{ij}(x, \alpha) \partial_{ij} - b_i(x, \alpha) \partial_i + c(x, \alpha) \) and \( \alpha = \frac{1}{2} \sigma^T \sigma \).

In the next section we justify the derivation of (1) by showing that \( u \) is the unique viscosity solution of (1), provided \( u \in C(\partial) \). Therefore we first have to show that \( u \in C(\partial) \); the following result is taken from P. L. Lions [28] (where much more general results are given). We will assume (for simplicity) that \( \partial \Omega = \Gamma_+ \cap \Gamma_0 \) where \( \Gamma_-, \Gamma_0 \) are disjoint, closed, possibly empty subsets of \( \partial \), and

\[
a_{ij}(x, \alpha) n_i(x) n_j(x) = 0, \quad b_i(x, \alpha) n_i(x) - a_{ij}(x, \alpha) \partial_{ij} d(x) \leq 0 \quad \text{on } \Gamma_+ \times \mathcal{A},
\]

(18)

where \( n \) is the unit outward normal and \( d(x) = \text{dist}(x, \partial) \). We will also assume that

\[
\exists w \in W^{1,\infty}(\partial), \quad A_{\alpha} w \leq f_{\alpha} \text{ in } \mathcal{O} \times \mathcal{A}, \quad w = \varphi \text{ on } \Gamma_0; \quad \forall \alpha \in \mathcal{A}, \quad w(\alpha) = \varphi \text{ on } \Gamma_0;
\]

(19)

\[
\exists C > 0, \forall (w, y) \in \partial \times \Gamma_0, \exists \mathcal{P}: J(w, \mathcal{P}) \leq \varphi(y) + C|x - y|.
\]

(20)

**Theorem II.1.** Under assumptions (18)-(20), \( u \in C^{0,\theta}(\partial) \) and \( u = \varphi \) on \( \Gamma_0 \); here \( \theta = \lambda/\lambda_0 \) if \( \lambda < \lambda_0 \), \( \theta \) arbitrary in \((0, 1)\) if \( \lambda = \lambda_0 \) and \( \theta = 1 \) if \( \lambda > \lambda_0 \) with \( \lambda_0 = \sup \{\text{Tr}(\partial_{ij} \sigma \partial_{ij} \sigma^T) + \partial_i b \cdot \xi/ \alpha \in \mathcal{A}, \alpha \in \mathcal{A}, |\xi| = 1\}. \)

**Remark II.1.** In P. L. Lions [38], it is shown that conditions (18)-(20) are very natural. Certain extensions are also given and examples of situations where (19)-(20) hold are indicated.

**Remark II.2.** Of course, if \( \partial = \mathbb{R}^N \), (18)-(20) are vacuous and (19) holds.
Remark II.3. The exponent $\theta$ given above is optimal as it is shown in the following example.

Example II.1. Take $\varnothing = \{-1, +1 \}$, $\sigma = 0$, $b(x, a) = x$, $f = 0$, $\varphi = 1$, $c = \lambda$. Then one checks easily that $u(x) = |x|^4$ and $\lambda_a = 1$. 

The proof of this result uses probabilistic and analytic arguments based on the dynamic programming principle.

We conclude this section by a result yielding one possible characterization of the value function in terms of maximum subsolution

**Theorem II.2.** The value function $u$ satisfies: $\sigma^T \cdot \nabla u \in L^2_{\text{loc}}(\varnothing)$ and

$$A_a u \leq f_a \text{ in } D'(\varnothing), \quad \forall a \in \mathcal{A};$$

i.e. $u$ is a subsolution of the HJB equation. In addition it is the maximum one in the sense that if $v \in C(\varnothing)$ satisfies (21) and $\lim_{d(x) \to 0} (v - u) \leq 0$ then $v \leq u$ in $\varnothing$.

**II.3. Viscosity solutions and HJB equations.** The following result shows that, boundary conditions being prescribed, the value function is the unique viscosity solution of (1).

**Theorem II.3.** If $u \in C(\varnothing)$, then $u$ is a viscosity solution of equation (1).

Conversely, if $v \in C(\overline{\varnothing})$ is a viscosity solution of (1) satisfying: $v = u$ on $\partial \varnothing$ then $v \equiv u$ in $\varnothing$.

This result, proved in P. L. Lions [38], justifies the derivation of the HJB equation and shows that (1) characterizes the value function.

**Sketch of proof.** The fact that $u$ is a viscosity solution of (1) is obtained by using the so-called *optimality principle* and by remarking that, taking advantage of the definition of viscosity solutions, one may replace $u$ by smooth test functions $\psi$ and thus one may perform on $\psi$ the ‘usual’ derivation of (1).

The converse statement is proved by probabilistic considerations and careful choices of test functions $\psi$.

**II.4. Further results.** In P. L. Lions [38], [40] these results are extended to more general situations and to other control problems (optimal stopping, time-dependent problems...). It is also possible to show that if (18)–(19) hold then one may restrict the infimum to admissible systems, where the probability space and the Brownian motion are fixed. Results
concerning the density of Markovian controls are also given. Let us finally mention that in P. L. Lions and M. Nisio [47], Theorem II.3 is used to derive a general uniqueness result for nonlinear semi-groups.

III. Regularity of the value function

III.1. Regularity results. In this section, we follow and extend the approach of N. V. Krylov [26], [27] concerning the verification of the HJB equation in a more usual sense. The idea is first to obtain some regularity result on $u$ and then to check (1). Since we do not want to impose non-degeneracy assumptions, we need to assume that $\partial \Theta = \Gamma_- \cap \Gamma_+$, where $\Gamma_-$, $\Gamma_+$ are closed disjoint, possibly empty subsets of $\partial \Theta$, and

$\exists \nu > 0, \quad \forall (x, a) \in \Gamma_+ \times \mathcal{A}$

either $a_{ij}(x, a) n_i(x) n_j(x) \geq \nu > 0$

or $a_{ij}(x, a) n_i(x) n_j(x) = 0, \quad b_i(x, a) n_i(x) - a_{ij}(x, a) \partial_{ij} d(x) \geq \nu.$

Write

$$\lambda_1 = \sup \{2 | \partial_i \sigma^T \cdot \xi_i|^2 + \text{Tr} (\partial_i \sigma \cdot \partial_i \sigma^T) + 2 \partial_i b \cdot \xi_i | x \in \mathbb{R}^N, a \in \mathcal{A}, |\xi| = 1\}.$$  

**Theorem III.1.** Under the assumptions (18), (22), if $\lambda > \lambda_1$ then the value function $u$ belongs to $W^{1, \infty}(\Theta)$ and satisfies $u = \varphi$ on $\Gamma_\cap \Gamma_+$ and

$u$ is semi-concave in $\Theta$, i.e.: $\exists C > 0, \partial^2_x u \leq C$ in $D'(\Theta), \forall |\xi| = 1.$

**Corollary III.1.** Under the assumptions of Theorem III.1, $u$ satisfies

$$A_a u \in L^\infty(\Theta) \quad \text{and} \quad \sup_{a \in \mathcal{A}} \|A_a u\|_{L^\infty(\Theta)} < \infty, \quad (24)$$

and the HJB equation holds a.e.:

$$\sup_{a} \{A_a u - f_a\} = 0 \quad \text{a.e. in } \Theta.$$

**Corollary III.2.** Under the assumptions of Theorem III.1, if there exist $p \in \{1, \ldots, N\}, \nu > 0,$ and an open set $\omega$ contained in $\Theta$ such that for all $x \in \omega$ we can find $n \geq 1, \theta_1, \ldots, \theta_n \in ]0, 1[, \alpha_1, \ldots, \alpha_n \in \mathcal{A}$ satisfying:

$$\sum_{i=1}^n \theta_i = 1, \quad \sum_{i=1}^n \theta_i a_{kl}(x, \alpha_i) \xi_k \xi_l \geq \nu \sum_{j=1}^p \xi_j^2 \quad \forall \xi \in \mathbb{R}^N,$$

then $\partial_{ij} u \in L^\infty(\omega)$ for $1 \leq i, j \leq p$.  ■
A slightly weaker form of these results was first given in P. L. Lions [42] and the above results appear in P. L. Lions [38] (see also [34]).

Sketch of proof. The estimate (23) implies (24) and Corollary III. 2 in a straightforward way, and the fact that the HJB equation holds a.e. is a direct consequence of (23)–(24) and the fact that the value function is a viscosity solution of (1). Next, the proof of (23) is obtained by new a priori estimates of two kinds; (i) a boundary estimate obtained by a p.d.e. device, (ii) an interior estimate obtained by a probabilistic method.

Remark III.1. In general, the assumption $\lambda > \lambda_1$ is necessary in order to obtain (23) as is shown by Example III.1: in this example $\lambda_1 = 2$ and $u(x) = |x|^4$; now observe that $u$ satisfies (23) if and only if $\lambda > \lambda_1$.

However, in the uniformly elliptic case, it is possible to assume only $\lambda \geq 0$

**Theorem III.2.** In the uniformly elliptic case, i.e., in the case where

$$\exists \nu > 0, \ a(x, a) \geq \nu I_N \quad \forall (x, a) \in \overline{\mathcal{C}} \times \mathcal{A}$$

we have $u \in W^{2, \infty}(\mathcal{O})$.

This result was first proved in P. L. Lions [43] with the additional assumption $\lambda > \lambda_1$; in [43] a new a priori estimate method was introduced. This method was simplified in L. C. Evans and P. L. Lions [21] and Theorem III.2 was proved in [21].

An additional regularity result was recently obtained by L. C. Evans [18], [19];

**Theorem III.3 (L. C. Evans).** Under the assumptions of Theorem III.2, $u \in C^{2, \theta}(\mathcal{O})$ for some $\theta \in ]0, 1[$.

### III.2. Uniqueness results.

In the proceeding section we obtained regularity results which ensure that the HJB holds a.e. One may ask if this yields a characterization of the value function. The following example shows that if $\tilde{u} \in W^{1, \infty}$ satisfies (24) and the HJB equation a.e., it need not be identical with $u$, as is shown by the following example:

**Example III.1.** Take $\sigma = 0$, $b = a$, $c = \lambda$, $f = 1$, $\mathcal{A} = \{a \in \mathbb{R}^N | |a| \leq 1\}$. Then clearly $u \equiv 1$, but for all $\beta$, $x_0 \in \mathbb{R}^+ \times \mathbb{R}^N$

$$\tilde{u}(x) = \frac{1}{\lambda} \left(1 - \beta \exp(-\lambda|x - x_0|)\right)$$

satisfies (24) and the HJB equation: $|D\tilde{u}| + \lambda \tilde{u} = 1$ everywhere except at $x_0$. 


Therefore we need some extra condition in order to characterize the value function. We have

\textbf{Theorem III.4.} Let \( \tilde{u} \in C(\overline{\mathcal{O}}) \cap W^{1,\infty}_{{\text{loc}}}(\mathcal{O}) \) satisfy: \( \tilde{u} = u \) on \( \partial \mathcal{O} \) and

\[ A_a \tilde{u} \leq f_a \quad \text{in} \quad \mathcal{D}'(\mathcal{O}), \quad \forall \alpha \in \mathcal{A}; \tag{21}\]

\[ \sup_{a \in \mathcal{A}} \{ A_a \tilde{u} - f_a \} = 0 \quad \text{in the sense of measures}; \tag{26}\]

\[ \exists g \in L^\infty(\mathcal{O}), \quad A \tilde{u} \leq g \quad \text{in} \quad \mathcal{D}'(\mathcal{O}). \tag{27}\]

Then \( \tilde{u} = u \) in \( \overline{\mathcal{O}} \).

\textit{Remark III.1.} Of course, if \( \mathcal{O} = \mathbb{R}^N \), no boundary conditions are needed. Instead, we assume, for example \( \tilde{u} \in C_b(\mathbb{R}^N) \).

Observe that condition (27) appears to be quite sharp since in Example III.1, \( (A \tilde{u})^+ \in L^p(\mathbb{R}^N) \) for all \( p < N(\in \mathcal{M}^N(\mathbb{R}^N)) \).

\textit{Sketch of the proof.} The proof of this Theorem has two ingredients: (i) the probabilistic \( L^p \)-estimates due to N. V. Krylov [26], (ii) careful bounds on the commutation of regularizing kernels and operators with variable coefficients.

\textbf{III.3. Further results.} In the bibliography we give various references concerning previous versions of Theorems II. 2–3 and the analogous treatment of related problems such as optimal stopping, impulse control, time-dependent problems, or other boundary conditions, and the numerical approximation of HJB equations... Let us also mention that in the case where \( \mathcal{O} = \mathbb{R}^N \), Theorem III.1 was obtained independently by N. V. Krylov [28], [29] and the author [41].

\textbf{IV. Monge–Ampère equations}

\textbf{IV.1. Relations with HJB equations.} Let us now explain how the Monge–Ampère equation

\[ \det(D^2 u) = g(x) \quad \text{in} \quad \mathcal{O}, \quad u \text{ convex on} \quad \overline{\mathcal{O}}, \quad u = \varphi \quad \text{on} \quad \partial \mathcal{O} \]

is related to HJB equations. This relation was discovered independently by B. Gaveau [24] and N. V. Krylov [30] and is given by the following algebraic observation: if \( A \) is an \( N \times N \) nonnegative symmetric matrix, then

\[ (\det A)^{1/N} = \inf \{ \text{Tr}(AB) | \ B \geq 0, \ B = B^T, \ \det(B) = 1/N^N \}. \]
Therefore the above equation is equivalent to the HJB equation:

$$\sup_{B \in \mathbb{R}^d} [-b_{ij} \partial_{ij} u] = -g^{1/2} \text{ in } \mathcal{O}, \quad u = \varphi \text{ on } \partial \mathcal{O}$$

where $B = (b_{ij})$, $\mathcal{B} = \{B \succeq 0, B = B^T, \det B = 1/N^N\}$. 

**IV.2. Existence and regularity results.** In differential geometry, the question of the existence of smooth convex hypersurfaces with various prescribed curvatures, such as e.g. the Gaussian curvature, leads to the following Monge–Ampère equations:

$$\det(D^2 w) = H(x, u, \nabla u) \text{ in } \mathcal{O}, \quad u \text{ convex on } \overline{\mathcal{O}}, \quad u = \varphi \text{ on } \partial \mathcal{O}, \quad (3)$$

where we assume, for instance, $\varphi \in W^{2,\infty}(\mathbb{R}^N)$, $H \in C^\infty(\partial \mathbb{R}^N)$, $\mathcal{O}$ being a bounded convex domain in $\mathbb{R}^N$ ($N \geq 2$).

We will assume that $H$ satisfies:

$$\forall R < \infty, \exists v > 0, \quad H(x, t, p \geq v > 0 \quad \text{for } x \in \overline{\mathcal{O}}, \quad |t| \leq R,$$

$$p \in \mathbb{R}^N; \quad (28)$$

and that there exists $w \in C(\overline{\mathcal{O}})$ satisfying the following inequality in Alexandrov sense (cf. [1], [12]) — or in viscosity sense —

$$\det(D^2 w) \geq H(x, \varphi, Dw) \text{ in } \mathcal{O}, \quad w \text{ convex in } \mathcal{O}, \quad w = \varphi \text{ on } \partial \mathcal{O}. \quad (29)$$

**Theorem IV.1.** Under assumptions (28)–(29), there exists a minimum solution of (3) in $C^\infty(\partial \mathcal{O}) \cap C(\overline{\mathcal{O}})$ satisfying: $u \geq w$ in $\overline{\mathcal{O}}$.

In addition, if $H$ is non-decreasing with respect to $t$, $u$ is the unique solution of (3). \[\blacksquare\]

**Remark IV.1.** Variants and extensions of this result are to be found in P. L. Lions [45]; in particular the verification of (29) is discussed by the use of the results of Bakelman [2].

Let us recall that the case of $H = H(x, t)$ was studied by Pogorelov [50] and that a complete proof of the existence of smooth solutions was first given by S. Y. Cheng and S. T. Yau [12] using geometric arguments. The proof of this theorem is given in P.L. Lions [45]; it is based on a new approximation method of (3) by problems in $\mathbb{R}^N$, relying on the idea of “penalizing the domain $\mathcal{O}$”. The approximated problem may then be solved by using the relations with HJB equations and the results of Section III.
Finally, uniform a priori estimates are derived by the use of the classical Pogorelov estimates [50] and Calabi estimates [9]. In conclusion, let us point out that, when $H = H(x, t)$, L. Caffarelli, L. Nirenberg and J. Spruck [8] recently showed that $u \in C^\infty(\Omega)$ and that using their method, one is able to show that, in the above result, if $w \in W^{1,\infty}(\Omega)$ then $u \in C^\infty(\bar{\Omega})$.

*Notes added in proofs:* We saw recently that Theorem III.3 has been obtained independently by N. V. Krylov (Izv. Mat. Ser. 46 (1982), pp. 487–523).

Furthermore, the $C^{2, \alpha}(\bar{\Omega})$ regularity has been obtained and applied to the Monge–Ampère equations independently by N. V. Krylov (Mat. Sbornik 120 (1983), pp. 311–330; Izv. Mat. Ser. 47 (1983), pp. 75–108) and by L. Caffarelli, J. J. Kohn, L. Nirenberg and J. Spruck.

**Bibliography**

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