

On the Work of Simon Donaldson

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In 1982, when he was a second-year graduate student, Simon Donaldson proved a result [1] that stunned the mathematical world. Together with the important work of Michael Freedman (described by John Milnor), Donaldson's result implied that there are "exotic" 4-spaces, i.e., 4-dimensional differentiable manifolds which are topologically but not differentiably equivalent to the standard Euclidean 4-space R^4 . What makes this result so surprising is that $n = 4$ is the only value for which such exotic n -spaces exist. These exotic 4-spaces have the remarkable property that (unlike R^4) they contain compact sets which cannot be contained inside any differentiably embedded 3-sphere!

To put this into historical perspective, let me remind you that in 1958 Milnor discovered exotic 7-spheres, and that in the 1960s the structure of differentiable manifolds of dimension ≥ 5 was actively developed by Milnor, Smale (both Fields Medalists), and others, to give a very satisfactory theory. Dimension 2 (Riemann surfaces) was classical, so this left dimensions 3 and 4 to be explored. At the last Congress, in Warsaw, Thurston received a Fields Medal for his remarkable results on 3-manifolds, and now at this Congress we reach 4-manifolds. I should emphasize that the stories in dimensions 3, 4, and $n \geq 5$ are totally different, with the low-dimensional cases being much more subtle and intricate.

Although I have highlighted the exotic 4-space as a spectacular corollary of the Freedman/Donaldson results, this is a by-product; their work is actually devoted to studying *closed* 4-manifolds. To such a 4-manifold, one associates standard topological invariants. In particular, for an *oriented* manifold, one gets a symmetric integer matrix of determinant ± 1 defined by the intersection properties of the 2-cycles (and depending on a choice of basis). Freedman showed that all such matrices can occur for topological 4-manifolds. Donaldson's result was that, among positive definite matrices, only those equivalent to the unit matrix can occur for differentiable 4-manifolds.¹ This is a severe restriction and shows that the differentiable and topological situations are totally different.

¹Actually, in [1] Donaldson restricted himself to simply-connected manifolds, but more recently he has succeeded in removing this restriction.

The surprise produced by Donaldson's result was accentuated by the fact that his methods were completely new and were borrowed from theoretical physics, in the form of the Yang-Mills equations. These equations are essentially a nonlinear generalization of Maxwell's equations for electro-magnetism, and they are the variational equations associated with a natural geometric functional. Differential geometers study connections and curvature in fibre bundles, and the Yang-Mills functional is just the L^2 -norm of the curvature. If the group of the fibre bundle is the circle, we get back the linear Maxwell theory, but for non-abelian Lie groups, we get a nonlinear theory. Donaldson uses only the simplest nonabelian group, namely $SU(2)$, although in principle other groups can and will perhaps be used.

Physicists are interested in these equations over Minkowski space-time, where they are hyperbolic, and also over Euclidean 4-space, where they are elliptic. In the Euclidean case, solutions giving the absolute minimum (for given boundary conditions at ∞) are of special interest, and they are called *instantons*.

Several mathematicians (including myself) worked on instantons and felt very pleased that they were able to assist physics in this way. Donaldson, on the other hand, conceived the daring idea of reversing this process and of using instantons on a general 4-manifold as a new geometrical tool. In this he has been brilliantly successful: he has unearthed totally new phenomena and simultaneously demonstrated that the Yang-Mills equations are beautifully adapted to studying and exploring this whole new field.

Of course, the use of differential equations in geometry is not new; the study of geodesics or minimal surfaces are classical examples. However, in these cases a solution of the differential equation (e.g., a minimal surface) is used as a geometrical object. Donaldson's use of instantons is quite different. I should explain that instantons as solutions of a minimization problem are not unique but typically depend on a finite number of continuous parameters, and it is the nonlinear space of these instanton parameters that Donaldson uses as a geometrical tool. The closest prior example of such an approach is the (linear) Hodge theory of harmonic forms. In fact, Hodge was directly motivated by Maxwell's equations, and instantons are a natural nonlinear generalization of harmonic forms. In the linear case the parameter space is of course linear and determined by its dimension, but in the nonlinear case there is much more information embodied in the parameter space, which is a topologically interesting manifold.

The success of Donaldson's program depends on having a thorough understanding of the analysis of the Yang-Mills equations. One needs existence, regularity, and convergence theorems, all of which are quite delicate, involving both local and global aspects. Fortunately, C. H. Taubes [6, 7] and K. Uhlenbeck [8, 9] have provided these analytical foundations, and so one can proceed to use instantons as an effective geometric tool. However, instantons cannot be bought off the shelf: to use them one has to understand and become involved with the full details of the analysis, and Donaldson has had to do this in order to put them to geometric use.

The Yang-Mills equations depend on fixing a background metric on the 4-manifold and, as in Hodge theory, Donaldson has to study the effect of varying the metric in order to derive results which depend only on the underlying manifold. Because of the nonlinearity, this is a more serious problem than in Hodge theory and great care is needed.

In fact, the Yang-Mills equations depend only on the conformal class of the metric and this conformal invariance is fundamental in physics where it implies the absence of a basic length scale. Analytically it is a source of difficulty making the equations a delicate border-line case where certain compactness arguments just fail, so that a sequence of instanton solutions can pick up Dirac delta functions in the limit. It is, however, just this delicate failure that Donaldson exploits geometrically: instead of the delta functions being regarded as undesirable singularities, they provide the key link between the 4-manifold and the instanton parameter space. One might say that the physicist's ambivalence to particles and fields is the essence of Donaldson's theory.

When Donaldson proved his first result it was by no means clear if this was some isolated case or whether instantons could be used more generally. Since then, however, Donaldson has, with great insight and skill, developed and exploited instantons with remarkable success. He has extended his results to the case of indefinite intersection matrices, providing further constraints on the topology of differentiable 4-manifolds. He has also, in the other direction, produced new invariants of 4-manifolds which can be used to distinguish smooth manifolds which are topologically equivalent. In particular, he has shown that complex algebraic surfaces (of complex dimension 2 and so of real dimension 4) appear to play a key role. In a very elegant paper [2] he proved an existence theorem which showed that, on an algebraic surface, instantons (or rather their parameter spaces) have a purely algebraic description, coinciding with what algebraic geometers call stable vector bundles. His new invariants can then be calculated algebraically and he used this [3] to exhibit two algebraic surfaces which are homeomorphic but not diffeomorphic. One of these surfaces is rational and his results strongly suggest that the rationality of an algebraic surface may be a differentiable property (it is *not* topological).

I indicated earlier that mathematicians had been working on the original physicists' problem of explicitly finding all instantons on Euclidean 4-space. In a short but decisive paper [4] Donaldson linked this problem with algebraic vector bundles on the complex projective plane (viewed as a compactification of $R^4 = C^2$). He also applied similar ideas [5] to solve a related but more difficult physical problem, that of magnetic monopoles. He proved the remarkably simple result that the parameter space of monopoles of magnetic charge k can be identified with the space of rational functions of a complex variable of degree k .

When Donaldson produced his first few results on 4-manifolds, the ideas were so new and foreign to geometers and topologists that they merely gazed in bewildered admiration. Slowly the message has gotten across and now Donaldson's ideas are beginning to be used by others in a variety of ways.

From what I have said you can see that Donaldson has opened up an entirely new area; unexpected and mysterious phenomena about the geometry of 4 dimensions have been discovered. Moreover, the methods are new and extremely subtle, using difficult nonlinear partial differential equations. On the other hand, this theory is firmly in the mainstream of mathematics, having intimate links with the past, incorporating ideas from theoretical physics, and tying in beautifully with algebraic geometry. It is remarkable and encouraging that such a young mathematician can understand and harness such a wide range of ideas and techniques in so short a time and put them to such brilliant use. It is an indication that mathematics has not lost its unity, or its vitality.

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