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On Some of the Mathematical Contributions of Gerd Faltings

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One of the recent great moments in mathematics was when Gerd Faltings revealed the circle of ideas which led him to a proof of the conjecture of Mordell ([**1**]; see also [**2**, **3**]).

The conjecture, marvelous in the simplicity of its statement, had stood as a goad and an elusive temptation for over half a century: it is even older than the Fields Medal! In modern language it takes the following form:

If K is any number field and X is any curve of genus > 1 defined over K, then X has only a finite number of K-rational points.

To get a feeling for our level of ignorance in the face of such questions, consider that, before Faltings, there was not a single curve X (of genus > 1) for which we knew this statement to be true for all number fields K over which X is defined!

Already in the twenties, Weil and Siegel made serious attempts to attack the problem. Siegel, influenced by Weil's thesis, used methods of diophantine approximation, to prove that the number of *integral* solutions to a polynomial equation $f(X, Y) = 0$ (i.e., solutions in the ring of integers of a number field K) is finite, provided that f defines a curve over K of genus > 0, or a curve of genus 0 with at least three points at infinity.

In his thesis, Weil generalized Mordell's theorem on the finite generation of the group of rational points on an elliptic curve, to abelian varieties of any dimension. Weil then hoped to use this finite generation result for the rational points on the jacobian of a curve to go on to show that when a curve of genus > 1 is imbedded in its jacobian, only a finite number of the rational points of the jacobian can lie on the curve. Not finding a way to do this, he decided to call his proof of finite generation (the "theorem of Mordell-Weil") a thesis, despite Hadamard's advice not to be satisfied with half a result!

After this work of Weil and Siegel there was little progress for thirty years. It was in the sixties and early seventies that several new developments occurred in algebraic geometry and number theory which were to influence Faltings (work of

Grothendieck, Serre, Mumford, Lang, Néron, Tate, Manin, Shafarevich, Parsin, Arakelov, Zarhin, Raynaud, and others).

These developments, which enter in an essential way in the work of Faltings, encompass three grand mathematical themes, and Faltings proved the conjecture of Mordell, by first establishing the truth of some other outstanding conjectures—fundamental to arithmetic and to arithmetic algebraic geometry. In the next few minutes I should like to try to convey some sense of the sweep of Faltings's accomplishment by touching on those themes, and those conjectures.

1. The arithmetization of geometry and the geometrization of arithmetic. Nowadays the analogy between number fields and fields of rational functions on algebraic curves (over finite fields) is so well imprinted upon our view of both number theory and the theory of algebraic curves that it is hard to imagine how we might deal with either theory, if deprived of that analogy. A firm understanding of the power of such analogies was present already in the work of Kronecker, and of Dedekind and Weber at the end of the last century. This understanding was deepened by the development of algebraic number theory, in the hands of Artin and Chevalley, of algebraic geometry, via the foundations developed by Zariski, Weil, and Serre, and more recently, of arithmetic algebraic geometry, whose foundations are given by the language of schemes, of Grothendieck.

In the language of schemes, a smooth curve over a finite field and the ring of integers in a number field are not merely analogous: they are two instances of the same notion (*regular schemes of dimension one, of finite type over* $\mathbf{Z}$). Similarly, a family of curves over a "base" curve over a finite field and a curve over the ring of integers in a number field are companion instances of the same notion.

This is not to say that this analogy is thoroughly understood! Why, it has only been relatively recently, thanks to the groundbreaking work of Arakelov, that we have begun to see a format for bringing the *archimedean places* of a number field into the geometric picture.

Moreover, this "synthetic view," extraordinarily efficacious for carrying problems and conjectures from the realm of function fields to the realm of number fields and back again, is far less satisfactory when it comes to carrying the *proofs* of those conjectures from one realm to the other.

For example, the analogue of Mordell's conjecture in the function field case was first settled by Manin back in 1963. A different proof was given by Grauert in 1965. Arakelov (using a beautiful idea of Parsin) found another proof in 1971. Yet another proof, also using Parsin, was given by Zarhin in 1974.

Even when we were armed with these approaches to the function field case, the number field case seemed intractable for almost a decade until Faltings discovered a method, analogous to that of Parsin-Zarhin, which established the classical conjecture of Mordell over number fields. To this day we lack number field analogues of the other approaches to the problem—say, of Manin's original proof, or of Arakelov's (Do they exist?). Judging from Faltings's published contributions to Arakelov's theory [4] one might imagine that he himself had been simultaneously pursuing two approaches to the Mordell conjecture in the number field case: one suggested by Arakelov's work, and the other by Zarhin's. The problem yielded, in 1983, to the second approach.

Faltings's method in the number field case, and Zarhin's in the function field case, was to settle Mordell's conjecture by first answering a more "geometric" question:

Kodaira had raised the problem of studying, or perhaps "classifying," all (truly varying) families of smooth curves of a given genus over a fixed (not necessarily complete) base curve. It was Shafarevich, at the 1962 International Congress, who first brought attention to the analogue of Kodaira's problem in arithmetic, and to its significance. One version of this analogue, now known as *Shafarevich's Conjecture for Curves*, states that

There are only a finite number of nonisomorphic curves of a given genus > 1 *defined over a fixed number field and possessing good reduction outside a fixed finite set of primes in the ring of integers of that number field.*

One way of paraphrasing Kodaira's original problem is by formulating the "function field analogue" of Shafarevich's conjecture, where the ring of integers in a number field is replaced by a base curve over a finite field. By an ingenious argument which happily worked as well in the number field case as in the function field case, Parsin had shown in 1968 that Mordell's conjecture follows from Shafarevich's conjecture.

Very roughly, Parsin's idea is as follows: Fix $g > 1$. Given a curve X of genus g and a rational point P on X over a number field K, Parsin produced a convering Y of X which is ramified only above P, and whose number field of definition and set of bad primes are "uniformly bounded" in terms of the data: g, the number field of definition of X and P, and the set of bad primes of X. Since Y determines the pair (X, P) up to finite ambiguity, it follows that if there are only finitely many such Y's (*Shafarevich's conjecture*) then there only finitely many such P's (*Mordell's conjecture*).

In their respective contexts, both Zarhin's and Faltings's attack on Mordell's conjecture is to prove Shafarevich's conjecture for curves (over function fields, and over number fields, respectively), and then to appeal to Parsin's idea.

2. Curves and abelian varieties. It was Weil, in his proof of the "Riemann hypothesis for curves over finite fields," who first made essential use of the passage from curves to abelian varieties to derive important consequences for the arithmetic of curves.

The "geometric" insight, that in pursuing questions about curves it sometimes pays to appeal to their jacobians, goes further back. Indeed, the fact that we see such a close relationship between curves and their jacobians is one of our many legacies from the Italian school of algebraic geometry.

With the trend towards the arithmetization of geometry, it was natural to study closely models for the jacobians of curves, or more generally for abelian varieties, over the rings of integers of number fields. Néron, in 1964, discovered the remarkable, and remarkably useful, fact that any abelian variety over a number field has a "best" model over the ring of integers of the number field—"best," from the point of view of niceness of the reduction of the model modulo prime ideals in the ring. Although N'eron, at the time he did his work, was not aware of Kodaira's contributions in the complex analytic case, one may view Néron's theory as a far-reaching amplification and arithmetization of a program initiated by Kodaira.

Néron models now play an important role in any close arithmetic study of abelian varieties, and in particular, they play a role in the detailed analysis of compactifications of moduli spaces for abelian varieties. The systematic *arithmetic* study of moduli spaces and their compactifications—a study initiated by the magnificent work of Mumford—in turn plays a key role in Faltings's approach. Compactification of moduli spaces of abelian varieties over **Z**, incidentally, is a subject to which Faltings has returned more recently: By refining, in certain respects, the work of Ching-Li Chai, Faltings has clarified some questions of arithmetic compactifications, and has thereby significantly simplified, and rendered more natural, the logical structure of his proof of Mordell's conjecture. In his initial proof [**1**] Faltings took a somewhat more circuitous route, using moduli spaces of curves rather than of abelian varieties (and the published account of the technical issues was supremely succinct, requiring a certain expertise on the part of the reader).

Thanks to the close relationship between curves and their jacobians (the classical theorem of Torelli plus a finiteness result concerning polarizations), Shafarevich's conjecture for curves (of genus > 1) reduces to a similar conjecture (also called *Shafarevich's conjecture*) for abelian varieties:

There are only a finite number of abelian varieties of fixed dimension over a fixed number field whose Néron models possess good reduction outside a fixed finite set of primes of the number field.

This conjecture was also settled affirmatively by the work of Faltings. It was settled in tandem with another basic arithmetic question:

3. Abelian varieties and Galois representations. It was by considering number-theoretic analogies of the classical conjectures of Hodge (concerning algebraic cycles) and geometric analogues of the conjecture of Birch and Swinnerton-Dyer that Tate in 1963 formulated the following conjecture, which links the problem of classifying abelian varieties (up to isogeny) to that of classifying the Galois representations to which they give rise:

Let l be a prime number. Let K be a number field, and $\overline{K}$ an algebraic closure of K. An abelian variety A over K is determined up to isogeny (over K) by the *natural representation of* $\mathrm{Gal}(\overline{K}/K)$ *on the* Q_l*-vector space*

$$V_l(A) = \mathrm{Hom}(Q_l, A_l(K)),$$

where $A_l(K)$ *denotes the group of* $\overline{K}$*-valued points of* A*, of* l*-power order.*

This clearly basic 'link' between abelian varieties (denizens of algebraic geometry) and Galois representations (ostensibly more 'elementary' creatures) was proved by Tate himself in 1967 for a finite field K, but over number fields it was not even known to be the case for elliptic curves, before the work of Faltings.

By a beautiful argument (which makes crucial use of two theories upon which we shall comment in a moment, and) which shuttles back and forth between the conjecture of Shafarevich for abelian varieties and the conjecture of Tate, Faltings showed that both of these conjectures are true. The phrase "shuttles back and forth" is quite inadequate to characterize Faltings's mode of argument, which captures in its weave all the mathematical themes upon which I have touched.

The "two theories" referred to are the *Theory of Heights* and the *Theory of p-Divisible Groups* (and, more generally, of *Group Schemes of Exponent* p).

The *Theory of Heights* was initiated by Weil in 1928 as a technique for "counting" rational points on abelian varieties and was used in an essential manner in his proof of the theorem "of Mordell-Weil." This theory was further developed by Néron, by Tate, and more recently was given a new twist in the work of Arakelov.

The *Theory of p-Divisible Groups* was invented by Serre and Tate, and, independently, by Barsotti in the mid-sixties to provide a technique to analyze the way in which p-power torsion points on abelian varieties "degenerate" when specialized to characteristic p. In 1966 Tate proved the analogue of his conjecture on abelian varieties for p-divisible groups over a local field of characteristic 0. Faltings uses this theorem, and, moreover, makes essential use of an important refinement of it (covering the case of group schemes of exponent p) due to Raynaud.

Even the above recitation does not completely exhaust the list of longstanding conjectures established by Faltings in the course of his work on the Mordell conjecture. For example, an important adjunct to the conjecture of Tate concerning the representations of the Galois group $\mathrm{Gal}(\overline{K}/K)$ on l-power torsion points of an abelian variety A defined over a number field K is the assertion of *semisimplicity* of the representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_l(A)$. This *Semisimplicity Conjecture* has also been proved by Faltings (as its "function field analogue" had been proved by Zarhin in the course of his work). Moreover the proof of semisimplicity plays a structural role in the proof of the other conjectures.

The above *Semisimplicity Conjecture* had been formulated by Grothendieck as the "dimension one case" of a more general question (semisimplicity of Galois representations acting on the d-dimensional l-adic cohomology of irreducible smooth projective varieties over number fields, for any d). One could find support for Grothendieck's conjecture, at the time he made it, in the work of Serre concerning the richness of the action of Galois on the torsion points of elliptic curves defined over number fields. Thanks to Faltings, we now know the *Semisimplicity Conjecture of Grothendieck* to be true for $d = 1$. This result of Faltings, incidentally, together with a technique coming from Faltings's proof, has very recently been used by Serre in a deep study of the action of Galois on torsion points of abelian varieties of arbitrary dimension g, defined over number fields.

The general case of Grothendieck's conjecture (i.e., for $d > 1$) is still open.

We have been discussing only Gerd Faltings's approach to the conjecture of Mordell, but his other mathematical contributions, whether they be concerned with moduli spaces of abelian varieties, the Riemann-Roch theorem for arithmetic surfaces, or p-adic Hodge theory, all immediately impress one as the work of a marvelously original mind from which we may expect similarly wonderful things in the future.

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