Proceedings of the International Congress of Mathematicians
Berkeley, California, USA, 1986

The Work of M. H. Freedman

JOHN MILNOR

Michael Freedman has not only proved the Poincaré hypothesis for 4-dimensional topological manifolds, thus characterizing the sphere S^4, but has also given us classification theorems, easy to state and to use but difficult to prove, for much more general 4-manifolds. The simple nature of his results in the topological case must be contrasted with the extreme complications which are now known to occur in the study of differentiable and piecewise linear 4-manifolds.

The "n-dimensional Pioncaré hypothesis" is the conjecture that every topological n-manifold which has the same homology and the same fundamental group as an n-dimensional sphere must actually be homeomorphic to the n-dimensional sphere. The cases $n = 1, 2$ were known in the nineteenth century, while the cases $n \geq 5$ were proved by Smale, and independently by Stallings and Zeeman and by Wallace, in 1960–61. (The original proofs needed an extra hypothesis of differentiability or piecewise linearity, which was removed by Newman a few years later.) The 3- and 4-dimensional cases are much more difficult.

Freedman's 1982 proof of the 4-dimensional Poincaré hypothesis was an extraordinary tour de force. His methods were so sharp as to actually provide a complete classification of all compact simply connected topological 4-manifolds, yielding many previously unknown examples of such manifolds, and many previously unknown homeomorphisms between known manifolds. He showed that a compact simply connected 4-manifold M is characterized, up to homeomorphism, by two simple invariants. The first is the 2-dimensional homology group

$$H_2 = H_2(M; Z) \cong Z \oplus \cdots \oplus Z,$$

together with the symmetric bilinear intersection pairing

$$\omega \colon H_2 \otimes H_2 \to Z.$$

This pairing, which is defined as soon as we choose an orientation for M, must have determinant ± 1 by Poincaré duality. The second is the Kirby-Siebenmann obstruction class, an element

$$\sigma \in H^4(M; Z/2) \cong Z/2$$

that vanishes if and only if M is stably smoothable. In other words, σ is zero if and only if the product $M \times R$ can be given a differentiable structure, or equivalently a piecewise linear structure. These two invariants ω and σ can be prescribed arbitrarily, except for a relation in one special case. If the form ω happens to be even, that is if $\omega(x, x) \equiv 0 \pmod 2$ for every $x \in H_2$, then the Kirby-Siebenmann obstruction must be equal to the Rohlin invariant:

$$\sigma \equiv \text{signature}(\omega)/8 \pmod 2.$$

(Freedman's original proof that these two invariants characterize M up to homeomorphism required an extra hypothesis of "almost-differentiability," which was later removed by Quinn.)

If the intersection form $\omega \neq 0$ is indefinite or has rank at most eleven, then it follows from known results about quadratic forms that M can be built up (nonuniquely) as a connected sum of copies of four simple building blocks, each of which may be given either the standard or the reversed orientation. One needs the product $S^2 \times S^2$, the complex projective plane CP^2, and two exotic manifolds which were first constructed by Freedman. One of these is a nondifferentiable analogue of the complex projective plane, and the other is the unique manifold whose intersection form ω is positive definite and even of rank eight. (This ω can be identified with the lattice generated by the root vectors of the Lie group E_8. As noted by Rohlin in 1952, a 4-manifold with such an intersection form can never be differentiable.) By way of contrast, if we allow positive definite intersection forms, then the number of distinct simply connected manifolds grows more than exponentially with increasing middle Betti number.

Freedman's methods extend also to noncompact 4-manifolds. For example, he showed that the product $S^3 \times R$ can be given an exotic differentiable structure, which contains a smoothly embedded Poincaré homology 3-sphere and hence cannot be smoothly embedded in euclidean 4-space [**11**, **16**]. His methods apply also to many manifolds which are not simply connected [**22**]. For example, a "flat" 2-sphere in 4-space is unknotted if and only if its complement has free cyclic fundamental group; and a flat 1-sphere in S^3 has trivial Alexander polynomial if and only if it bounds a flat 2-disk in the unit 4-disk whose complement has free cyclic fundamental group.

The proofs of these results are extremely difficult. The basic idea, which had been used in low dimensions by Moebius and Poincaré, and in high dimensions by Smale and Wallace, is to build the given 4-manifold up inductively, starting with a 4-dimensional disk, by successively adding handles. The essential difficulty, which does not arise in higher dimensions, occurs when we try to control the fundamental group by inserting 2-dimensional handles, since a 2-dimensional disk immersed in a 4-manifold will usually have self-intersections. This problem was first attacked by Casson, who showed how to construct a generalized kind of 2-handle with prescribed boundary within a given 4-manifold. Freedman's major technical tool is a theorem which asserts that every Casson handle is actually homeomorphic to the standard open handle, (closed 2-disk) $\times$ (open 2-disk). The proof involves a delicately controlled infinite repetition argument in the spirit of the Bing school of topology, and is nondifferentiable in an essential way.

PAPERS OF M. H. FREEDMAN

1. *Automorphisms of circle bundles over surfaces*, Geometric Topology, Lecture Notes in Math., vol. 438, Springer-Verlag, 1974, pp. 212–214.

2. *On the classification of taut submanifolds*, Bull. Amer. Math. Soc. **81** (1975), 1067–1068.

3. *Uniqueness theorems for taut submanifolds*, Pacific J. Math. **62** (1976), 379–387.

4. *Surgery on codimension* 2 *submanifolds*, Mem. Amer. Math. Soc. No. 119 (1977).

5. *Une obstruction élémentaire à l'existence d'une action continue de groupe dans une variété* (with W. Meeks), C. R. Acad. Sci. Paris Ser. A **286** (1978), 195–198.

6. Λ*-splitting* 4*-manifolds* (with L. Taylor), Topology **16** (1977), 181–184.

7. *A geometric proof of Rochlin's theorem* (with R. Kirby), Proc. Sympos. Pure Math, vol. 32, part 2, Amer. Math. Soc., Providence, R.I., 1978, pp. 85–98.

8. *Remarks on the solution of first degree equations in groups*, Algebraic and Geometric Topology, Lecture Notes in Math., vol. 664, Springer-Verlag, 1978, pp. 87–93.

9. *Quadruple points of* 3*-manifolds in* S^4, Comment. Math. Helv. **53** (1978), 385–394.

10. *A converse to (Milnor-Kervaire theorem)* $\times R$ *etc....*, Pacific J. Math. **82** (1979), 357–369.

11. *A fake* $S^3 \times R$, Ann. of Math. **110** (1979), 177–201.

12. *Cancelling* 1*-handles and some topological imbeddings*, Pacific J. Math. **80** (1979), 127–130.

13. *A quick proof of stable surgery* (with F. Quinn), Comment. Math. Helv. **55** (1980), 668–671.

14. *Planes triply tangent to curves with nonvanishing torsion*, Topology **19** (1980), 1–8.

15. *Slightly singular* 4*-manifolds* (with F. Quinn), Topology **20** (1981), 161–173.

16. *The topology of* 4*-manifolds*, J. Differential Geom. **17** (1982), 357–454 (see also F. Quinn, ibid, p. 503).

17. *A surgery sequence in dimension four; the relations with knot concordance*, Invent. Math. **68** (1982), 195–226.

18. *Closed goedesics on surfaces* (with J. Hass, P. Scott), Bull. London Math. Soc. **14** (1982), 385–391.

19. *A conservative Dehn's Lemma*, Low Dimensional Topology, Contemp. Math., vol. 20, Amer. Math. Soc., Providence, R.I., 1983, pp. 121–130.

20. *Imbedding least area incompressible surfaces* (with J. Hass, P. Scott), Invent. Math. **71** (1983), 609–642.

21. *Homotopically trivial symmetries of Haken manifolds are toral* (with S.-T. Yau), Topology **22** (1983), 179–189.

22. *The disk theorem for* 4*-dimensional manifolds*, Proc. Internat. Congr. Math. Warsaw 1983, vol. 1, Pol. Sci. Publ., 1984, pp. 647–663.

23. *There is no room to spare in four-dimensional space*, Notices Amer. Math. Soc. **31** (1984), 3–6.

24. *Atomic surgery problems* (with A. Casson), Four-Manifold Theory, Contemp. Math., vol. 35, Amer. Math. Soc., Providence, R.I., 1984, pp. 181–199.

OTHER REFERENCES

25. H. Poincaré, *Cinquième complément á l'Analysis situs* (1904), Oeuvres 6, Paris, 1953, pp. 435–498.

26. L. Siebenmann, *Amorces de la chirurgie en dimension 4, un* $S^3 \times R$ *exotique* (d'apres A. Casson et M. H. Freedman), Séminaire Bourbaki (1978–79), Exp. No. 536, Lecture Notes in Math., vol. 770, Springer-Verlag, 1980, pp. 183–207.

27. ——, *La conjecture de Poincaré topologique en dimension* 4 (d'apres M. H. Freedman), Seminaire Bourbaki (1981–82), Exp. No. 588, Astérisque, no. 92–93, Soc. Math. France, Paris, 1982, pp. 219–248.

INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540, USA