Milnor $K$-Theory and Galois Cohomology

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Let $F$ be a field and $m$ a natural number, $(m, \text{char } F) = 1$. It follows from the exact sequence $1 \to \mu_m \to F_{\text{sep}}^* \xrightarrow{m} F_{\text{sep}}^* \to 1$ and the Hilbert Theorem 90 that there is the natural isomorphism $l: F^*/F^{*m} \to H^1(F, \mu_m)$. The cup-product in the cohomology theory of groups gives the homomorphism

$$F^* \otimes F^* \otimes \cdots \otimes F^* \to H^n(F, \mu_m^{\otimes n}),$$

satisfying the Steinberg condition and therefore giving the homomorphism:

$$\alpha_{n,m}: K_n(F)/mK_n(F) \to H^n(F, \mu_m^{\otimes n}),$$

by the formula [10]:

$$\alpha_{n,m}(\{a_1, a_2, \ldots, a_n\}) = l(a_1) \cup l(a_2) \cup \cdots \cup l(a_n),$$

where $K_n(F)$ is the Milnor group of $F$. This homomorphism is called the norm residue map of degree $n$.

There is a conjecture that all $\alpha_{n,m}$ are isomorphisms for any $F$. If $n = 1$ it is a consequence of the Hilbert Theorem 90. In the case $n = 2$ this conjecture was proved in [5]. If $n \geq 3$ the answer is known only for some fields.

The theorem stating the bijectivity of $\alpha_{2,m}$ has many applications. The present paper is devoted to some applications in the Brauer group theory.

If $\mu_m \subseteq F$, then

$$H^2(F, \mu_m^{\otimes 2}) = H^2(F, \mu_m) \otimes \mu_m = m\text{Br}(F) \otimes \mu_m,$$

where $m\text{Br}(F)$ is the subgroup of elements of exponent $m$ in the Brauer group $\text{Br}(F)$ of $F$. In this case the norm residue map of degree 2 coincides with the homomorphism

$$K_2(F)/mK_2(F) \to m\text{Br}(F) \otimes \mu_m,$$

introduced in [9].

1. Cyclic algebras. Let $F$ be a field and $G$ the Galois group of $F_{\text{sep}}/F$. We denote by $X(F)$ the character group $\text{Hom}_c(G, \mathbb{Q}/\mathbb{Z})$ of $G$. For any character $\chi \in X(F)$ let $F(\chi)/F$ be the field extension corresponding to $\ker \chi \subseteq G$ by the Galois theory. It is clear that $F(\chi)/F$ is the cyclic extension of degree $n$ which equals the order of $\chi$ in $X(F)$. The Galois group of this extension has the natural generator $\sigma_\chi$, uniquely determined by the equality $\chi(\sigma_\chi) = 1/n + \mathbb{Z}$. 

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Let \( a \in F^* \), \( \chi \in X(F) \), \( n = (F(\chi): F) \). We define the structure of an \( F \)-algebra on an \( n \)-dimensional vector space \( A = A(a, \chi) \) over \( F(\chi) \) with basis \( u_0, u_1, \ldots, u_{n-1} \) by the following formula:

\[
(xu_i) \cdot (yu_j) = \begin{cases} 
  x \cdot \sigma^i(y) \cdot u_{i+j}, & i + j < n, \\
  a \cdot x \cdot \sigma^i(y) \cdot u_{i+j-n}, & i + j \geq n.
\end{cases}
\]

\( A(a, \chi) \) is a central simple algebra over \( F \) of dimension \( n^2 \) and is said to be a cyclic algebra of degree \( n \) [1].

There is a conjecture that \( \text{Br}(F) \) is generated by the classes \([A(a, \chi)]\) of cyclic algebras. We can formulate this conjecture more precisely: for any \( m \in \mathbb{N} \) the group \( m \text{Br}(F) \) is generated by \( A(a, \chi), a \in F^*, \chi \in mX(F) \).

In the following cases the last conjecture has been proved:

1. If \( (m, \text{char } F) = 1 \) and \( \mu_m \subset F \). In this case the conjecture is equivalent to the surjectivity of the norm residue map \( \alpha_{2,m} \). Indeed, the element \( \alpha_{2,m}(\{a, b\}) \) equals \([A(a, \chi_b)] \otimes \xi_m \in m \text{Br}(F) \otimes \mu_m \), where \( \chi_b \) is the unique character, such that \( F(\chi_b) = F(b^{1/m}), \sigma_\chi(b^{1/m}) = \xi_m \cdot b^{1/m} \), and any character \( \chi \in mX(F) \) equals \( \chi_b \) for some \( b \in F^* \).

2. If \( m \) is a prime number and \( (F(\mu_m): F) < 3 \). In particular, the conjecture is proved if \( m = 2, 3 \) [5, 6].

3. If \( m = 4 \) [5].

4. If \( m = p^k \) and \( p = \text{char } F \) [1].

There are the following relations in \( \text{Br}(F) \) between the classes of cyclic algebras:

1. \([A(ab, \chi)] = [A(a, \chi)] + [A(b, \chi)]\),
2. \([A(a, \chi + \rho)] = [A(a, \chi)] + [A(a, \rho)]\),
3. \([A(a, \chi)] = 0 \) if \( a \) is a norm in \( F(\chi)/F \).

There is a conjecture that any relation between the classes \([A(a, \chi)]\) with \( a \in F^*, \chi \in mX(F) \) in \( m \text{Br}(F) \) is a consequence of (1)–(3). This conjecture has been proved in the following cases:

1. \((m, \text{char } F) = I \) and \( \mu_m \subset F \). In this case the conjecture is equivalent to the injectivity of the norm residue map \( \alpha_{2,m} \).

2. If \( m \) is a prime number and \( (F(\mu_m): F) \leq 2 \). In particular, the conjecture is proved if \( m = 2, 3 \).

3. If \( m = p^k \) and \( p = \text{char } F \).

We can take both conjectures together and consider the following version of the norm residue map \( \alpha_{2,m} \). Let \( S_m(F) \) be the factorgroup \((F^* \otimes mX(F))/B\) where \( B \) is the subgroup generated by \( a \otimes \chi \), where \( a \) is a norm in \( F(\chi)/F \).

The correspondence \( a \otimes \chi \mapsto [A(a, \chi)] \) defines the following homomorphism: \( \beta_m: S_m(F) \to m \text{Br}(F) \). The first conjecture is equivalent to the surjectivity of \( \beta_m \) and the second conjecture to the injectivity of \( \beta_m \).

If \( \mu_m \subset F \), then the homomorphism \( \beta_m \otimes \mu_m \) is equal to the norm residue map \( \alpha_{2,m} \) and therefore is an isomorphism.

One can show that to prove the bijectivity of \( \beta_m \) it is sufficient to construct the corestriction map \( S_m(L) \to S_m(F) \) for a finite extension \( L/F \). So far we can...
define such a map only for some quadratic extensions. This gives the proof in the case \((F(\mu_m): F) \leq 2\).

2. Generators and relations in Brauer group. The problem is to find a set of generators and relations in \(\text{Br}(F)\). It is sufficient to study the \(p\)-primary component \(\text{Br}(F)\{p\}\) of \(\text{Br}(F)\) for any prime \(p\). We assume that \(p \neq \text{char} F\).

Let \(k \in \mathbb{N}, q = p^k, F_k = F(\mu_q)\). For \(x \in F_k^*\) and \(\chi \in qX(F_k)\) let \([x, \chi] = \text{cor}([A(x, \chi)]) \in q\text{Br}(F_k)\), where \(\text{cor}: \text{Br}(F_k) \to \text{Br}(F)\) is a corestriction map [3]. We know that \(q\text{Br}(F_k)\) is generated by \([A(x, \chi)]\). There is a question whether \(q\text{Br}(F)\) is generated by \([x, \chi], x \in F_k^*, \chi \in qX(F_k)\).

A simple example: \(F = \mathbb{R}, p = 2, k \geq 2\) shows that the answer is negative in general. Unexpectedly the answer is positive in almost all other cases.

We suppose that \(F_k/F\) is a cyclic extension (if, for example, \(p\) is odd).

The following case we call exceptional: \(p = 2, F_k/F\) is a quadratic extension, and \(\tau(\xi_q) = \xi_q^{-1}\), where \(\tau\) is a generator of \(\text{Gal}(F_k/F)\). (The example above belongs to the exceptional case.) The case which is not exceptional is called general. When \(p\) is odd or \(\sqrt{-1} \in F, p = 2\) we have the general case.

**THEOREM [7].** 1. In the general case \(q\text{Br}(F)\) is generated by \([x, \chi], x \in F_k^*, \chi \in qX(F_k)\).

2. In the exceptional case \(q\text{Br}(F)\) is generated by \([x, \chi], x \in F_k^*, \chi \in qX(F_k)\) with the classes of quaternion algebras

\[
\langle a, \rho \rangle \overset{\text{def}}{=} [A(a, \rho)], \quad a \in F^*, \rho \in 2X(F).
\]

If \(F_k/F\) is not a cyclic extension (this is possible only if \(p = 2\)) the situation is more complicated. There are some arguments in [7], showing that in this case the group \(q\text{Br}(F)\) does not have a "simple set" of generators.

We have the following relations between \([x, \chi], x \in F_k^*, \chi \in qX(F_k)\):

1. \([xy, \chi] = [x, \chi] + [y, \chi]\),
2. \([x, \chi + \eta] = [x, \chi] + [x, \eta]\),
3. \([x, \chi] = 0\) if \(x\) is a norm in \(F_k(\chi)/F_k\),
4. \([\tau(x), \tau(\chi)] = [x, \chi], \) where \(\tau \in \text{Gal}(F_k/F)\).

**THEOREM [7].** In the general case any relation between \([x, \chi], x \in F_k^*, \chi \in qX(F)\) is a consequence of (1)–(4).

In the exceptional case we have the following additional relations between \([x, \chi], x \in F_k^*, \chi \in qX(F_k)\), and the classes of quaternion algebras \(\langle a, \eta \rangle, a \in F^*, \eta \in 2X(F)\):

5. \((ab, \eta) = (a, \eta) + (b, \eta)\),
6. \((a, \eta + \rho) = (a, \eta) + (a, \rho)\),
7. \((a, \rho) = 0\) if \(a\) is a norm in \(F(\rho)/F\),
8. \([x, \chi] = [N_{F_k/F}(x), \chi], x \in F_k^*, \chi \in 2X(F)\),
9. \([a, \chi] = 0\), where \(a \in F^*\) and \(F_k(\chi) = F_k(b^{1/m})\) for some \(b \in F^*\).
THEOREM [7]. In the exceptional case any relation between \([x, x], x \in F_k^*, \chi \in qX(F_k), \text{ and } \langle a, \eta \rangle, a \in F^*, \eta \in 2X(F)\), is a consequence of (1)-(9).

Studying our generators \([x, x]\) in detail we can prove the following statement:

THEOREM [7].
1. In the general case \(q\text{Br}(F)\) is generated by the classes of algebras of dimension not more than \(q^{2p^{k-1}}\).

2. In the exceptional case \(q\text{Br}(F)\) is generated by the classes of algebras of dimension not more than \(q\).

COROLLARY [6]. The group \(p\text{Br}(F)\) is always generated by the classes of algebras of dimension \(p^2\).

3. Brauer group as an abstract group. It is a well-known fact that any Brauer group is abelian and torsion. In [2] A. Brumer and M. Rosen conjectured that if \(2 \cdot \text{Br}(F)\{p\} \neq 0\) then \(\text{Br}(F)\{p\}\) contains a nontrivial divisible subgroup.

THEOREM [6]. Let \(p\) be an odd number, \((F(\xi_p) : F) \leq 3\), and \(\text{Br}(F)\{p\} \neq 0\). Then \(\text{Br}(F)\{p\}\) contains a nontrivial divisible subgroup.

This theorem proves the conjecture of A. Brumer and M. Rosen for \(p = 3\) and any field.

In the case \(p = 2\) a nontrivial group \(\text{Br}(F)\{p\}\) may not contain a divisible subgroup. For example, any elementary abelian 2-group is isomorphic to \(\text{Br}(F)\{2\}\) for some field \(F\). The situation is different if \(\text{Br}(F)\) contains an element of order 4.

THEOREM [6]. If \(2 \cdot \text{Br}(F)\{2\} \neq 0\), then \(\text{Br}(F)\{2\}\) contains a nontrivial divisible subgroup.

In [4] B. Fein and M. Schacher conjectured that any divisible abelian torsion group is a Brauer group for some field. This conjecture is proved in [8].

REFERENCES


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