Geometrical Aspects of Representation Theory

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I learned from I. M. Gel’fand that Mathematics of any kind is a Representation theory...

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(from the short address on occasion of Professor Gel’fand’s 70th birthday)

1. Geometry of primitive ideals.

1.1. Let $G$ be a complex algebraic group with Lie algebra $g$. I would like to have a geometric picture for the structure of primitive ideals of the enveloping algebra $U(g)$. The only universal answer known so far is the “orbit method,” saying that primitive ideals should correspond to coadjoint orbits in the dual $g^*$ of $g$. Sometimes this principle doesn’t work, however. The situation may be improved if one looks not just onto an orbit itself but on a certain “resolution data” over it as well. I propose to consider diagrams like this:

$$
\begin{array}{ccc}
\mu^{-1}(G \cdot \lambda) & \leftrightarrow & T^\lambda X \\
G \cdot \lambda & \leftrightarrow & g^* \\
& \pi & X = G/P
\end{array}
$$

(1.1.1)

Here $\lambda \in g^*$, $P$ is a connected algebraic subgroup of $G$ with Lie algebra $p$, such that $\lambda\llbracket p,p \rrbracket = 0$, and $T^\lambda X := G \times_P (\lambda + p^\perp)$ is a “twisted cotangent bundle” on $X$ (i.e., an affine bundle with symplectic structure on the total space, having Lagrangian fibres). The map $\mu$ is the “moment mapping,” taking $(g, \xi) \in G \times (\lambda + p^\perp)$ to $(\text{Ad } g) \cdot \xi$. If $p$ is a good polarization of $\lambda$ so that the orbit method applies, then $\mu$ is an isomorphism of $T^\lambda X$ onto the orbit $G \cdot \lambda$ and $\pi \cdot \mu^{-1}(\lambda + p^\perp)$ is a single point of $G/P$.

**Definition 1.1.2.** A solvable subgroup $P$, as above, is called a polarizing subgroup if the set $\pi \cdot \mu^{-1}(\lambda + p^\perp) \subset X$ consists of a finite number of $P$-orbits.

Let $F_x = G \cdot \lambda \cap \text{Ad } x \cdot (\lambda + p^\perp)$, $x \in X$. Given a polarization $p$ (see [Di]), one gets a Lagrangian fibering: $G \cdot \lambda \to X$ with fibres $F_x$. There is no such fibering for a general polarizing subgroup $P$. Yet, one has

**Proposition 1.1.3.** $\{F_x, x \in X\}$ is a $G$-invariant Lagrangian family in $G \cdot \lambda$. □
Various $F_x$ may intersect each other. However, after replacing $G \cdot \lambda$ by \( \mu^{-1}(G \cdot \lambda) \) , the family \( \{F_x\} \) turns into the "resolved" one: \( \tilde{F}_x := \mu^{-1}(G \cdot \lambda) \cap T_x^\lambda X \), induced by the standard family of fibres of the cotangent bundle.

**Proposition 1.1.4.** \( \mu^{-1}(G \cdot \lambda) \) is a coisotropic subvariety of \( T^\lambda X \) and its projection onto \( G \cdot \lambda \) coincides with the reduction map (relative to the 0-foliation on a coisotropic subvariety). \( \Box \)

1.2. Given \( P \), a polarizing subgroup, one can construct a finite number of \( g \)-modules. Set \( F_0 = G \cdot \lambda \cap (\lambda + p^{-}) \). It follows from Propositions 1.1.3 and 1.1.4 that \( \mu^{-1}(F_0) \) is a Lagrangian subvariety of \( T^\lambda X \). Let \( \Lambda \) be one of its irreducible components and \( \pi: \Lambda \to X \), the projection. Clearly, \( \pi(\Lambda) \) is a \( P \)-stable irreducible subvariety of \( X \). By Definition 1.1.2, there is a unique open orbit \( P \cdot w \subset \pi(\Lambda), (w \in X) \), and \( \Lambda \) may be viewed as a "twisted conormal bundle" to \( P \cdot w \). The space of distributions supported on \( P \cdot w \) (with a certain twist depending on \( \lambda \)) is a \( U(g) \)-module, via the action of \( g \) on \( X \). Let \( L_{\lambda,p,A} \) be its \( U(g) \)-submodule generated by the zero-order \( P \)-eigendistributions, i.e., those having no derivatives in directions normal to the orbit.

**Theorem 1.2.1.** Given \( \lambda \in g^* \), one can find a polarizing subgroup \( P \) such that all \( L_{\lambda,p,A} \) are nontrivial irreducible \( U(g) \)-modules (for all irreducible components \( \Lambda \)). \( \Box \)

In particular, every \( \lambda \in g^* \) has a polarizing subgroup.

1.3. From now on \( G \) is assumed to be a connected reductive group and \( g^* \) is identified with \( g \) via the Killing form. Let \( B \) be a Borel subgroup of \( G \), \( T \) a maximal torus of \( B \), \( W \) the Weyl group of \( (G,T) \), \( X = G/B \) the flag manifold, \( n \) the nilpotent radical of Lie \( B \), and \( \rho \) half sum of the roots in \( g/\text{Lie} B \).

We remark that \( B \) is a polarizing subgroup for all points of Lie \( B \), since the number of \( B \)-orbits in \( G/B \) is finite. Thus, the machinery of §§1.1–1.2 applies. The case of a nilpotent orbit \( G \cdot \lambda \) is most interesting. The moment map \( \mu \) of (1.1.1) then turns into the Springer resolution:

\[ \mu: \text{the usual cotangent bundle } T^*X \to N = \{n \in g|\text{ad} \, n \text{ is nilpotent}\} \] (1.3.1)

Here is the translation of (1.1.2)–(1.2.1) into our present language:

Each component of \( G \cdot \lambda \cap n \) is a Lagrangian subvariety of \( G \cdot \lambda \); \( \mu^{-1}(n) \) is the union of all conormal bundles \( T_{X_w}^\ast X \) to the Bruhat cells \( X_w = B \cdot w \cdot B/B \) \( (w \in W) \); \( \mu^{-1}(G \cdot \lambda \cap n) \) equals \( T_{X_w}^\ast X \) for some \( w \in W \);

The closure of an irreducible component of \( \mu^{-1}(G \cdot \lambda \cap n) \) equals \( L_{\lambda,B \cdot A} \) for some \( A \in W \);

If \( \tilde{A} = T_{X_w}^\ast X \), then \( L_{\lambda,B \cdot A} = L_w \) is the irreducible highest weight module with the highest weight \( -\rho - w \cdot \rho \).
the flag manifold $X$, filtered by degree, as usual. The characteristic variety of a $\mathcal{D}_X$- (resp. $U(g)$)-module is a subvariety of $T^*X$ (resp. $g^*$). It is known, for instance, that $S(\mathcal{D}_X \otimes_{U(g)} L_w)$ is a Lagrangian subvariety of $T^*X$ and $S(L_w)$ is its image under the moment map $\mu$ [BB].

Let $I_w = \text{Ann } L_w$ be a primitive ideal of $U(g)$. If $G = \text{SL}_n$, the particularly nice geometric classification of $I_w$'s in terms of characteristic varieties is expected:

$$S(\mathcal{D}_X \otimes L_w) = T^\ast_{X_w}X \quad \text{(conjectural)}; \quad (1.4.1)$$
$$S(\mathcal{D}_X/\mathcal{D}_X \cdot I) = G \cdot T^\ast_{X_w}X \quad \text{(conjectural)}; \quad (1.4.2)$$
$$I_y \subset I_w \Leftrightarrow S(\mathcal{D}_X/\mathcal{D}_X \cdot I_y) \supset S(\mathcal{D}_X/\mathcal{D}_X \cdot I_w) \quad \text{(conjectural)}; \quad (1.4.3)$$
$$S(U(g)/I_w) = G \cdot (n \cap w \cdot n \cdot w^{-1}) \quad \text{(Joseph).} \quad (1.4.4)$$

All these statements but (1.4.3) are false whenever $G$ is simple $\neq \text{SL}_n$. In general, one knows only that $S(U(g)/I_w)$ is the closure of a nilpotent orbit [BB, KT, Gi2].

2. Representations of Weyl groups and Hecke algebras.
2.1. Why is it interesting to study the objects named in the title? The first reason is that irreducible representations of a Weyl group $W$ have a beautiful geometric description, discovered by Springer [Spr], connecting them with nilpotent orbits in $g^*$ (as in the “orbit method”!). Secondly, nilpotent orbits arise also as characteristic varieties of annihilators of simple $g$-modules. So, combining with Springer, one gets a strange relation: simple $g$-modules $\leftrightarrow$ (simple) $W$-modules. That relation turns out to be crucial both in Joseph's work on primitive ideals and in the work of Kazhdan-Lusztig [KL1]. Let me mention, as an illustration, that to finite-dimensional irreducible $g$-modules one attaches the “sign-representation,” appearing already in the Weyl character formula.

Another source of our interest originates from the observation that the Hecke algebra $H(q)$ may be viewed as a $q$-analogue of the group-algebra $\mathbb{Z}[W]$. Yet no Springer-type geometric theory for $H(q)$-modules was known so far. Such a theory [Gi3] is given below for affine Hecke algebras together with a conjectural one for the ordinary Hecke algebra $H(q)$ (the “affine” case appears to be simpler). We begin with decomposing the two-sided regular representation of $H(q)$ or $\mathbb{Z}[W]$ into so-called “two-sided cell representations” (cf. §2.4). These cell representations of $H(q)$ are expected to coincide with those introduced by Kazhdan-Lusztig [KL1], while in the Weyl group case they provide an alternative simple approach to the Springer theory (cf. [Gi2]). Next, each two-sided cell representation corresponds to a nilpotent conjugacy class and irreducible constituents of such a representation form an “$L$-packet,” parametrized by irreducible representations of a certain (often abelian) finite group. It should be emphasized that although the algebra $H(q)$ is (unnaturally) isomorphic to $\mathbb{C}[W]$ the cell representations for $H(q)$ and $\mathbb{C}[W]$ do not correspond to each other when $q \to 1$. Thus, there are two similar, but different, geometric pictures (for $q = 1$, resp. $q \neq 1$). I guess that the picture for $q \neq 1$, rather than Springer's one, is relevant for the representation theory of complex Lie groups. Furthermore, it
is likely to be a correspondence: $G$-modules $\leftrightarrow H(q)$-modules, similar to that known for reductive groups over finite fields (cf. §3) with $H(q)$ playing the role of an algebra of intertwining operators.

2.2. Keeping to the notations of §1.3, let $\tilde{N} = T^*X$ and $\Lambda = \tilde{N} \times_N \tilde{N} = \{(x_1, x_2) \in \tilde{N} \times \tilde{N} | \mu(x_1) = \mu(x_2)\}$. The group $C^* \times G$ acts on $N$, $\tilde{N}$, and $\Lambda$ ($C^*$ by multiplication and $G$ by conjugation). Let $K_{C^* \times G}(\Lambda)$ be the Grothendieck group of $C^* \times G$-equivariant coherent $\mathcal{O}_\Lambda$-sheaves.

We regard $\Lambda$ as a correspondence from $\tilde{N}$ to $\tilde{N}$. Clearly, $\Lambda \circ \Lambda = \Lambda$. There is a ring structure on $K_{C^* \times G}(\Lambda)$ arising from composition of “sheaf-valued correspondences.” That is, given $[F], [F'] \in K_{C^* \times G}(\Lambda)$, let $[F] \cdot [F']$ be the alternating sum of cohomology sheaves of the complex:

$$F \cdot F' = (p_{13})_*(p^*_{12}F \otimes_{C^* \times G} p^*_{23}F'),$$

where $p_{ij}: \tilde{N} \times \tilde{N} \times \tilde{N} \to \tilde{N} \times \tilde{N}$ is the projection along the factor not named.

Thus, $K_{C^* \times G}(\Lambda)$ becomes a $\mathbb{Z}[q, q^{-1}]$-algebra with $\mathbb{Z}[q, q^{-1}]$-action arising from that of the representation ring $R(C^*) = \mathbb{Z}[q, q^{-1}]$.

For a maximal torus $T$ of $G$ let $\text{Hom}(T, C^*)$ be the lattice of weights and $\tilde{W} := W \rtimes \text{Hom}(T, C^*)$, the semidirect product (the “affine Weyl group”). Let $H$, resp. $\tilde{H}$, be the Hecke algebras attached to $W$, resp. $\tilde{W}$ (cf. [KL1]).

**Theorem 2.2.1** [Gi3]. There is an algebra isomorphism $\tilde{H} \simeq K_{C^* \times G}(\Lambda)$. □

When viewed as a subvariety of $\tilde{N} \times \tilde{N} \cong T^*(X \times X)$, $\Lambda$ becomes the union of conormal bundles to all $G$-orbits in $X \times X$. Hence, $\Lambda$ has $\#W$ irreducible components, all of the same dimension $2d = 2 \cdot \dim X$. The group $CH_{2d}(\Lambda)$ of $2d$-dimensional cycles in $\Lambda$ is generated by these components. It also has a ring structure, so that the assignment $F \mapsto "2d\text{-dimensional part of supp } F"$ gives rise to a ring homomorphism $\text{supp}: K_G(\Lambda) \to CH_{2d}(\Lambda)$. Theorem 2.2.1 yields the following commutative diagram of ring homomorphisms:

\[
\begin{array}{ccc}
H & \xrightarrow{i} & \tilde{H} & \xrightarrow{(2.2.1)} & K_{C^* \times G}(\Lambda) \\
\downarrow q & & \downarrow q & & \downarrow \text{forgetting} \\
\mathbb{Z}[W] & \xrightarrow{i} & \mathbb{Z}[\tilde{W}] & \xrightarrow{(2.2.2)} & \mathbb{Z}[\tilde{W}] \\
\downarrow \text{id} & & \downarrow r & & \downarrow \text{supp} \\
\mathbb{Z}[W] & \xrightarrow{\tau} & \mathbb{Z}[W] & \xrightarrow{\text{supp}} & CH_{2d}(\Lambda)
\end{array}
\]

where the embeddings $i$ are induced by the natural inclusion $W \hookrightarrow \tilde{W}$ and the projection $\tau$ is induced by the homomorphism $\tilde{W} \to W$, taking $\text{Hom}(T, C^*)$ to 1.

2.3. Consider the projection $\mu_\Lambda: \Lambda = \tilde{N} \times_N \tilde{N} \to N$. For a $G$-stable closed subvariety $Y \subset N$ let $\tilde{H}(Y)$ (resp. $\mathbb{Z}[\tilde{W}](Y)$ or $\mathbb{Z}[W](Y)$) be the two-sided ideal of $\tilde{H} = K_{C^* \times G}(\Lambda)$ (resp. $\mathbb{Z}[\tilde{W}]$ or $\mathbb{Z}[W]$) generated by sheaves or cycles supported on $\mu_\Lambda^{-1}(Y)$. Given a nilpotent conjugacy class $C \subset N$, set: $\tilde{H}(C) = \tilde{H}(\overline{C})/\tilde{H}(\overline{C} - C)$, $\mathbb{Z}[\tilde{W}](C) = \mathbb{Z}[\tilde{W}](\overline{C})/\mathbb{Z}[\tilde{W}](\overline{C} - C)$, etc. These
spaces are bimodules over the algebras in question. The following conjecture is very close to that of [Lu3].

**Conjecture 2.3.1.** The $\tilde{H}(C)$'s are precisely the two-sided cell representations of $\tilde{H}$ in the sense of [KL1].

Next, identify $H$ with a subalgebra of $\tilde{H}$ and set

$$H(C) = \tilde{H}(C) \cap H / \tilde{H}(C - C) \cap H.$$  

It is likely that these are precisely the two-sided cell representations of $H$.

**Conjecture 2.3.2.** $H(C) \neq 0 \iff C$ is a "special" orbit (in the sense of Lusztig).

If true, this would provide a geometric understanding of "special" orbits. I also think that the reason for the characteristic variety $S(\mathcal{D}_X \otimes \mathcal{L}_w)$ to be greater than $T_{X_w}^* X$ is this: it might be no $\mathbb{C}^* \times G$-equivariant sheaves contained in $H \subset \tilde{H}$ and supported on a single conormal bundle at the same time.

2.4. The center of $\mathbb{C} \otimes \tilde{H}$ is known to be isomorphic to $\mathbb{C}[T]^W [g, q^{-1}]$, where $\mathbb{C}[T]$ denotes the regular ring of $T$, a maximal torus. Let $I_{t,h}$ be the maximal ideal of the center, corresponding to a point $m = (t, h) \in \mathbb{C}^* \times T$ and let $\tilde{H}(t, h) := \mathbb{C} \otimes \tilde{H} / I_{t,h} \cdot \tilde{H}$ be the specialization of $\tilde{H}$ at $m = (t, h)$.

Let $\mu$: $\tilde{N}^m \to N^m$ be the restriction of the moment map: $\tilde{N} \to N$ to $m$-fixed point subvarieties. For $n \in N^m$, the fibre $X_n^h := \mu^{-1}(n) \subset N^m$ may be viewed as a variety of all Borel subgroups of $G$, containing both $h$ and exp $n$. The group $Z_G(h, n)$ acts on it by conjugation, giving rise to an action of the finite group $A(h, n) = Z_G(h, n) / Z^0_G(h, n)$ on $H_*(X_n^h)$.

**Proposition 2.4.1** [Gi3, KL3]. There is an $\tilde{H}(t, h)$-action on $H_*(X_n^h)$, commuting with that of $A(h, n)$. □

This $\tilde{H}(t, h)$-action is a $q$-analogue of a Weyl group action, defined by Springer [Spr].

**Proposition 2.4.2** [Gi3]. There exists an $A(h, n)$-invariant symmetric form $(\cdot, \cdot)$ on $H_*(X_n^h)$, which is "contravariant" with respect to the $\tilde{H}(t, h)$-action; i.e., $(a \cdot x, y) = (x, a^* \cdot y)$ for $x, y \in H_*(X_n^h)$ and an anti-involution: $a \mapsto a^*$ on $\tilde{H}$, taking $T_w$ to $T_{w^{-1}}$ when $w \in W \subset \tilde{W}$.

The form $(\cdot, \cdot)$ is defined by intersecting cycles $x, y \in H_*(X_n^h)$ in an ambient (smooth) variety $S$, the inverse image of a transversal slice to an orbit in $N^m$.

It follows from Propositions 2.4.1 and 2.4.2 that there is an orthogonal decomposition:

$$H_*(X_n^h) = \bigoplus K_{h, n, \chi} \otimes \chi,$$  

where $\chi$ runs over irreducible representations of $A(h, n)$ and $K_{h, n, \chi}$ are certain $\tilde{H}(t, h)$-modules.

2.5. The representation theory of $\tilde{H}(t, h)$ that I am going to present amazingly resembles that of highest weight g-modules (i.e., category $O$ of [BGG]). In both cases, three types of objects play the prominent role: simple modules $L_\alpha$, ...
their projective covers $P_{\alpha}$, and the intermediate “standard” modules $K_{h,n,\chi}$ (cf. (2.4.3)), which are the counterparts of the Verma modules. The $\tilde{H}$-module theory has an advantage of giving way to direct geometric interpretation. The module $K_{h,n,\chi}$, for instance, lives in homology of $X_n^h$ and has a contravariant form $(\cdot, \cdot)$, arising from intersecting cycles (cf. Proposition 2.4.2). Just as in the $g$-module case, all simple $\tilde{H}$-modules turn out to be of the form: $L_{\alpha} = K_{h,n,\chi}/\text{rad}(\cdot, \cdot)$, for some triple $(h, n, \chi)$, which is unique up to conjugation by $G$. So we write: $\alpha = (h, n, \chi)$. The multiplicities $(K_{\alpha} : L_{\beta})$ and $(P_{\alpha} : L_{\beta})$ are in both cases related to each other by similar formulas (cf. Proposition 2.5.4 and [BGG]), and can be expressed in terms of the intersection cohomology (Kazhdan-Lusztig type formula; cf. §§2.5.4–2.6.2 and [BK, BeBe]). Furthermore, the derived category of graded $\tilde{H}(t, h)$-modules is equivalent to the derived category of mixed complexes on $N^m$ (cf. Theorem 2.6.3 and [BG]) so that grading on modules corresponds to the weight filtration on sheaves. This is an $\tilde{H}$-counterpart of the well-known equivalence between $g$-modules and perverse sheaves (on the flag manifold), established via $D$-modules [BeBe, BK].

Fix $m = (t, h)$, a semisimple element of $C^* \times G$. Let $M(m) = \{ Z_G(h)\text{-conjugacy classes of } \alpha = (h, n, \chi); (\text{Ad } h) \cdot n = t^{-1} \cdot n, \text{ with } \chi \text{ an irreducible representation of } A(h, n) \text{ occurring in } H_*(X^h_n) \}$.

For $\alpha = (h, n, \chi) \in M(m)$ the group $A(h, n)$ may be viewed as a quotient of $\pi_1(O_{\alpha})$, where $O_{\alpha} = Z_G(h) \cdot n$ is an orbit in $N^m$. Let $L_\alpha$ be the intersection cohomology complex on $O_\alpha$ with the monodromy $\chi$. By the decomposition theorem [BBD], applied to the projective map $\mu: \tilde{N}^m \rightarrow N^m$, we get (cf. [BM]):

$$R\mu_*(C_\tilde{N}^m) = \bigoplus_{\alpha \in M(m)} L_\alpha \otimes L_\alpha \quad \text{(shifts disregarded)}, \quad (2.5.1)$$

where $L_\alpha$ are certain vector spaces. It turns out that: $L_\alpha \cong K_\alpha/\text{rad}(\cdot, \cdot)$.

Let $H_*(\Lambda^m, C)$ be the Borel-Moore homology of the $m$-fixed point subvariety $\Lambda^m \subset \Lambda$. Theorem 2.2.1 and (2.5.1) yield

**Proposition 2.5.2 [G13].** There are algebra isomorphisms:

$$\tilde{H}(t, h) \cong H_*(\Lambda^m, C) \cong \bigoplus_{i \geq 0, \alpha, \beta} \text{Hom}(L_\alpha, L_\beta) \otimes \text{Ext}^i(L_\alpha, L_\beta). \quad \square$$

**Theorem 2.5.3 [G13] (cf. [KL3]).** Suppose $t$ is not a root of unity. Then $\{L_\alpha | \alpha \in M(m), m = (t, h)\}$ is precisely the collection of all simple $\tilde{H}(t)$-modules. \quad \square

Given two orbits $O_\beta \subset O_\alpha$, $\beta = (h, n, \chi)$, consider the $\chi$-component of the stalk $(\mathcal{Y}^i L_\alpha)_n$, and set $L_{\alpha, \beta}^i = \dim(\mathcal{Y}^i L_\alpha)_n$.

Let $\mathcal{L}$ be the unipotent matrix with entries $L_{\alpha, \beta} = \sum_i L_{\alpha, \beta}^i$, $D$ the diagonal matrix: $D_{\alpha, \alpha} = \text{Euler characteristic of } O_\alpha$ (with compact support), and $(K : L)$, $(P : L)$ the multiplicity matrices with entries $(K_{\alpha} : L_{\beta})$, resp. $(P_{\alpha} : L_{\beta})$.

**Proposition 2.5.4.** $(K : L) = \mathcal{L}$ and $(P : L) = \mathcal{L} \cdot D \cdot \mathcal{L}^t$. \quad \square
2.6. Following an idea of Jantzen, one deforms the parameter \((h, n, x) = \alpha\) of a standard module in order to get a module \( \tilde{K}_\alpha \) over \( C[u] \), the polynomial ring, so that \( K_\alpha = \tilde{K}_\alpha / u \cdot \tilde{K}_\alpha \). The contravariant form on \( K_\alpha \) lifts to a \( C[u] \)-valued form on \( \tilde{K}_\alpha \). The Jantzen filtration \( J^\cdot \) on standard modules is defined by:

\[
J^k K_\alpha = \{ x \in \tilde{K}_\alpha | (x, \tilde{K}_\alpha) C u^k \cdot \tilde{K}_\alpha \}/(u).
\] (2.6.1)

Again, the construction has a direct geometric interpretation via \( C^* \)-equivariant cohomology (recall that \( \tilde{H}^*(BC^*) \cong \tilde{H}^*(\mathbb{CP}^\infty) \cong C[u] \)). The Jantzen filtration then turns out to be related to intersecting with a "hyperplane section." The Jantzen filtration on Verma modules is related, in the same way, to the monodromy on vanishing cycles, thus supporting the analogy [De] between the monodromy and the Lefschetz operators.

**Theorem 2.6.2 [Gi3]**. \( \text{Gr}_L K_\alpha \) (see (2.6.1)) is a semisimple \( \tilde{H}(t, h) \)-module and \( (\text{Gr}_L^i K_\alpha : L_\beta) = \mathcal{L}^{-i}_{\alpha, \beta}, i = 0, 1, \ldots \) \( \square \)

Consider the grading on \( \tilde{H}(t, h) \) given by the index "i" in the direct sum decomposition in Proposition 2.5.2. Using the assignment \( \mathcal{L}_\alpha \mapsto P_\alpha \) one obtains:

**Theorem 2.6.3 [BG]**. The category of mixed perverse sheaves on \( N^m \), having their Jordan-Hölder factors among \( \mathcal{L}_\alpha, (\alpha \in \mathcal{M}(m)) \), is equivalent to the category of bounded complexes:

\[
\cdots \rightarrow P_i \rightarrow P_{i+1} \rightarrow \cdots
\]

where \( P_i \) denotes a graded projective \( \tilde{H}(t, h) \)-module generated by elements of degree \( i \).

2.7. Given \( \alpha = (h, n, \chi) \in \mathcal{M}(m) \) let \( C = G \cdot n \). The action of \( \tilde{H} \) on \( L_\alpha \) gives rise to an algebra homomorphism (notation of §2.3): \( \tilde{H}(C) \rightarrow \text{End}(L_\alpha) \), vanishing on \( \tilde{H}(C-C) \). Let \( f: H(C) \hookrightarrow \tilde{H}(C) \rightarrow \text{End}(L_\alpha) \) be the induced homomorphism. The algebra \( H(n, h, \chi) := C \otimes f(H(C)) \) doesn’t change when \( (t, h) \in C^* \times G \) varies in a connected component of the isotropy subgroup of \( n \), for it is semisimple, hence rigid. Thus, \( H(n, h, \chi) \) depends only on the conjugacy class of \( (n, \overline{h}, \chi) \), where \( \overline{h} \) is the image of \( (t, h) \) in \( \pi_1(C) \) and \( \chi \) is viewed as an irreducible representation of its centralizer in \( \pi_1(C) \).

**Conjecture 2.7.1**. \( H(n, h, \chi) \) is a simple algebra.

Moreover, the assignment \( (n, \overline{h}, \chi) \mapsto H(n, h, \chi) \) (for \( n \) special, cf. Conjecture 2.3.2) seems to provide a parametrization of simple \( H \)-modules.

3. Towards the "Langlands geometry." Throughout this section \( G \) stands for a connected split reductive group with connected center and \( G^* \) for the dual group in the sense of [La].

3.1. Let \( \mathbb{F}_q \) be a finite field. The classification of irreducible complex representations of the finite group \( G(\mathbb{F}_q) \) [Lu2] splits into two parts. Given a simple \( G(\mathbb{F}_q) \)-module, one attaches to it a semisimple conjugacy class in \( G^*(\mathbb{F}_q) \) and a "unipotent" representation of a smaller reductive group [Lu2]. The classification of unipotent representations was carried out in [Lu2]. As presented there,
It appears to me quite mysterious. A part of it, at least, has a simple geometric explanation. Namely, consider irreducible representations arising from decomposition of the "principal series representation" $V = \mathbb{C}[G(F_q)/B(F_q)]$, where $B(F_q)$ is a split Borel subgroup. The ring $\text{Hom}_{G(F_q)}(V, V)$ is spanned by the double-cosets relative to $B(F_q)$ and is isomorphic to the Hecke algebra $\mathbb{C} \otimes H(q)$. Thus, the irreducible constituents of $V$ are in (1-1)-correspondence with simple $H(q)$-modules and the latter were (conjecturally) classified in §2.7.

3.2. Let $K$ be a p-adic field with the residue class field $F_q$. By the general "Langlands philosophy" [La], irreducible representations of $G(K)$ should correspond to conjugacy classes of homomorphisms: $\text{Gal}(\overline{K}/K) \rightarrow G^*(\mathbb{C})$, where $\text{Gal}(\overline{K}/K)$ denotes the Galois-Weil group. Following [Lu1], one considers the refined data $\alpha = (C, \chi)$, consisting of such a conjugacy class $C$ and an irreducible character $\chi$ of $\pi_1(C)$.

Let $\Gamma$ be the quotient of $\text{Gal}(\overline{K}/K)$, corresponding to the maximal tamely ramified extension of $K$. The group $\Gamma$ has the generators $F$ (Frobenius) and $M$ (Monodromy), subject to the relation: $F \cdot M \cdot F^{-1} = M^q$. Hence, the refined data attached to a homomorphism $\Gamma \rightarrow G^*(\mathbb{C})$ is determined by a triple $\alpha = (h, g, \chi)$, such that $h \cdot g \cdot h^{-1} = g^q$ ($h, g \in G^*(\mathbb{C})$). Deligne and Langlands conjectured (cf. [Lu1]) that:

Irreducible representations of $G(K)$ occurring in the "unramified principal series" are in (1-1)-correspondence with the conjugacy classes of triples $\alpha = (h, g, \chi)$ with $h$ semisimple and $g$ unipotent. (3.2.1)

As in §3.1, the irreducible representations in question are in (1-1)-correspondence with simple modules over $\tilde{H}(q)$, the "affine" Hecke algebra, attached to $G^*(\mathbb{C})$. Thus, the conjecture (3.2.1) follows from Theorem 2.5.3 (with $g$ replaced by $n = \log g$).

I believe that a similar mechanism is responsible for the general Langlands conjecture. Namely, there should be a variety $\tilde{G}^*$, a pure perverse sheaf $\mathcal{C}$ on $\tilde{G}^*$ and a proper map $\mu: \tilde{G}^* \rightarrow \text{Hom}(\text{Gal}(\overline{K}/K), G^*(\mathbb{C}))$ so that irreducible representations of $G(K)$ correspond to the irreducible constituents of $R\mu_*\mathcal{C}$ (cf. (2.5.1)). Furthermore, an appropriate derived category of admissible $G(K)$-modules is expected to be equivalent to a derived category of mixed constructible complexes on $\text{Hom}(\text{Gal}(\overline{K}/K), G^*(\mathbb{C}))$.

REFERENCES


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