Algebraic Geometry and Representation Theory

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0. Let $G$ be a reductive group over $\mathbb{Z}$. For any field $F$ we can consider the group $G_F$ of $F$-points on $G$. At first glance, the groups $G_F$ for different fields $F$ appear to have little in common with each other. I. Gelfand has conjectured that

1. The structure of the representations of $G_F$ has fundamental features which do not depend on a choice of $F$.

2. Moreover, it is possible to define representations by formulas which are universally valid over any local or finite fields.

In the book [GGP-S] both conjectures are proved for $G = SL_2$ (see Chapter 2, §§4.1 and 5.4). Unfortunately the second conjecture is not known for any other group.

Langlands reformulated the first conjecture in a more precise form [B], and it has been proven in a number of cases.

Since almost nothing is known about the second conjecture, I will postpone its discussion until the very end of this paper and will restrict myself to applications of Algebraic Geometry to Representation Theory.

1. Finite fields. Let $F$ be a finite field of characteristic $p$, $G$ a reductive $F$-group, and $G = G_F$. Let $B = T_0U \subset G$ be a Borel subgroup. For any character $\chi$ of $T_0$, we denote by $\pi_\chi$ the induced representation $\pi_\chi = \text{Ind}_{B_0}^G \chi$. Such representations are called "the principal series representations." We say that a character $\chi$ is generic if $\chi^w \neq \chi$ for any nontrivial element $w$ of the Weyl group $W$ of $G$. It is easy to see that

1. for any generic character $\chi: T_0 \to \mathbb{C}^*$, the induced representation $\pi_\chi$ is irreducible;

2. the representations $\pi_\chi, \pi_{\chi'}$ are equivalent if and only if $\chi, \chi'$ are conjugate under the action of the Weyl group $W$;

3. for any generic $t_0 \in T_0$ we have $\text{Tr} \pi_\chi(t_0) = \sum_{w \in W} \chi(t_0^w)$.

Macdonald has conjectured that for any Cartan subgroup $T \subset G$ and any generic character $\chi: T \to \mathbb{C}^*$, there exists a unique irreducible representation $\pi_\chi$ of $G$, such that for any generic $t \in T$ we have $\text{Tr} \pi_\chi(t) = \pm \sum_{w \in W_T} \chi(t^w)$, where
$W_T = \text{Norm}_G(T)/T$. Springer [Sp1] suggested a formula for the character of $\pi_X(g)$ for arbitrary $g$ in $G$. To formulate his conjecture we assume that $p$ is sufficiently large. Let $\mathfrak{g}$ be the Lie algebra of $G$, $\Omega \subset \mathfrak{g}$ a regular conjugacy class containing an element of the Lie algebra of $T$, and $\phi_\Omega$ the Fourier transform of the characteristic function of $\Omega$. Springer [Sp2] proved that the restriction of $\phi_\Omega$ to the set of nilpotent elements in $\mathfrak{g}$ does not depend on the choice of $\Omega$; he conjectured that $\text{Tr} \pi_X(u) = \phi_\Omega(\log u)$ for any unipotent $u$ in $G$. This conjecture was proved a little later [K]. Both Springer's proof of $\Omega$-independence and the proof of his conjecture are based on the algebro-geometric reduction to the case of principle series, where $T = T_0$.

At first glance, the formula for $\text{Tr} \pi_X$ is given piecemeal—one formula for unipotent elements, a completely different formula for semisimple elements. But Lusztig realized the existence of a natural algebro-geometric object describing $\text{Tr} \pi_X$. He showed that there exists a unique irreducible perverse sheaf $\mathcal{F}_X$ over $G$ such that for any $g$ in $G$ we have that $\text{Tr} \pi_X(g)$ equal to the trace of the Frobenius at $g$ on the fiber of $\mathcal{F}_X$ at $g$. Later he showed that the characters of all irreducible representations of $G$ can be expressed in terms of perverse sheaves.

Another application of Algebraic Geometry to representation theory of finite Lie groups is the realization of representations of $G$ in cohomology groups of algebraic varieties. It was discovered by Drinfeld (in the case $G = GL_n$) and by Deligne and Lusztig [D-L] (in the general case). This discovery led Lusztig ultimately to the classification of irreducible representations of reductive groups over a finite field [L].

2. Local fields. In the case when $F = \mathbb{R}$ or $\mathbb{C}$, the theory of $D$-modules [V] gave the possibility of finding characters of all irreducible representations of $G$. In the case when $F$ is a nonarchimedean field, $K$-theory provides the way to construct all the irreducible components of the unramified principle series [K-L]. This topic will be discussed in more details in V. Ginzburg's talk (see his preprint [G]).

Another application of Algebraic Geometry to representation theory of local groups is the work of Henniart (see his letter to Laumon) in which he proves the (weak) form of the local Langlands conjecture for $GL_n$. The proof is based on Laumon's theory of the Fourier transform of representations of Galois groups of local fields with positive characteristic.

3. Global fields. Let $F$ be a global field of finite characteristic $p$. In this case, $F$ is the field of rational functions on a curve $C$ over a finite field $\mathbb{F}_q$. Let $\mathfrak{A}$ be the ring of adèles of $F$ and let $\mathcal{O} \subset \mathfrak{A}$ be the subring of integral adèles. A special case of Langlands' conjecture states the existence of a natural one-to-one correspondence $\rho \rightarrow \pi_{\rho}$ between the set of equivalence classes of irreducible $n$-dimensional $l$-adic representations of $\pi_1(C)$, $l \neq p$, and the set of equivalence classes of unramified cuspidal automorphic representations of $GL_n(\mathfrak{A})$. This correspondence was established by V. Drinfeld for $n = 2$ [D] in the following way.
Let $\rho: \pi_1(C) \to GL_2(Q_1)$ be an irreducible representation. Drinfeld associates to $\rho$ a perverse sheaf $\mathfrak{F}_\rho$ on the space $\mathcal{M}_2$ of 2-dimensional vector bundles over $C$. The trace of the Frobenius on $\mathfrak{F}_\rho$ defines a function $\phi_\rho$ on the set $\mathcal{M}_2(F_q)$. Since $\mathcal{M}_2(F_q) \simeq GL_2(O) \backslash GL_2(A)/GL_2(F)$ (see [W]) we may consider $\phi_\rho$ as a function on $GL_2(A)/GL_2(F)$. Drinfeld showed that $\phi_\rho$ is an automorphic forms which corresponds to $\pi_\rho$.

Another application of Algebraic Geometry to automorphic forms in positive characteristic is the proof of the Ramanujan conjecture for automorphic representations of $GL_n(A)$ with at least one supercuspidal component.

4. Different fields. Until now we have discussed different types of fields separately. But they are very much related. For example, the notion of a unipotent representation for groups over finite fields [D-L] has a very precise analogue for representations of real groups [B-V].

5. A possibility. We were describing applications of Algebraic Geometry to Representation Theory. Now we try to envisage the possible algebro-geometric structure of Representation Theory. We will restrict ourselves to the case where $F$ is a local field. Let $G$ be a reductive group over $Z$.

It is difficult to imagine the existence of "an object" $\hat{G}$ over $Z$ such that irreducible representations of $G_F$ correspond to "$F$-points" of $\hat{G}$ for local fields $F$. Even in the simplest case $G = G_m$, the corresponding functor $F \to \{\text{Multiplicative characters of } F\}$ does not have an obvious algebro-geometric interpretation.

On the other hand, we can try to describe "series of representations." Let $H$, $G$ be quasisplit reductive $Z$-groups and let $\phi: ^LH \to ^LG$ be a morphism of the corresponding $L$-groups (see [B]). Langlands' philosophy predicts the existence of a correspondence between representations of $H_F$ and $G_F$ for any (local) $F$. We can try to imagine a "materialization" of this correspondence in the following way.

Does there exist a $Z$-variety $X$, a differential form $\omega$ of highest degree on $X$, and rational functions $R^\phi$, $K^\phi$ on $H \times G \times X \times X$ such that for any local field $F$ and a nontrivial additive character $\psi$ on $F$, the operators $T(h \times g)$ on $L^2(X_F, |\omega|_F)$, given by the formula

$$T(h \times g)(f)(x) \overset{\text{def}}{=} \int_{y \in X_F} \left( \frac{\psi(K^\phi(h, g, x, y))}{|R^\phi(h, g, x, y)|} \right) f(y)|\omega|_F,$$

define a projective unitary representation of $H_F \times G_F$ which realizes the Langlands correspondence? An example of such a construction is in [GGP-S]. In this case $G = SL_2$, and $H$ is the group of units in a quadratic extension $E$ of $Q$. We take $X = E$, $\omega$ to be a standard two form on $X$ ($\simeq A^2$), $R \equiv 1$, and define $K$ by the formula

$$K(h, g, x, y) = e^{-1}(aN(y) + dN(x) - Tr hxy)$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $N: E^* \to Q^*$ and $\text{tr}: E \to Q$ are the norm and the trace on $E$. It is natural to ask for which series of representations it is possible to realize Gelfand's conjecture in such a form.
REFERENCES


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