Uniformity and Irregularity

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Introduction. The purpose of this report is to draw attention to some non-trivial connections between the (“continuous”) theory of irregularities of distribution and discrete mathematics. The object of the theory of irregularities of distribution is to measure the uniformity (or nonuniformity) of sequences and point sets. For instance: how uniformly can an arbitrary set of $N$ points in the unit cube be distributed relative to a given family of “nice” sets (e.g., boxes with sides parallel to the coordinate axes, rotated boxes, balls, all convex sets). This theory lies on the border of many branches of mathematics (number theory, discrete geometry, combinatorics, etc.) and has very important applications, e.g., in numerical integration. Here we focus, of course, on the combinatorial aspects of the theory.

As an illustration, we shall discuss first two problems of a discrete nature which have fascinating connections with this “continuous” theory. The first question is concerned with balanced two-colorings of finite sets in a square (the problem is due to G. Tusnády). Let $\mathcal{P} = \{p_1, \ldots, p_N\}$ be a distribution of $N$ points in the unit square $[0,1]^2$. Let $f: \mathcal{P} \to \{-1, +1\}$ be a “two-coloring” of $\mathcal{P}$. Let $B$ denote any rectangle in $[0,1]^2$ with sides parallel to the coordinate axes (an aligned rectangle, in short). Consider the function

$$T(N) = \sup_{\mathcal{P}} \inf_f \sup_{B} \left| \sum_{p_i \in \mathcal{P} \cap B} f(p_i) \right|,$$

where the supremum is taken over all subsets $\mathcal{P} \subset [0,1]^2$, $\# \mathcal{P} = N$, and all aligned rectangles $B$ in $[0,1]^2$, and the infimum is taken over all “two-colorings” $f$ of $\mathcal{P}$. Tusnády conjectured that $N^c > T(N) \to \infty$. A positive answer was obtained in [1]:

$$c_1 (\log N)^4 > T(N) > c_2 \log N.$$

The proof of the lower bound was nonconstructive. Recently Roth [19] investigated the following explicit construction: let

$$\mathcal{P}_\alpha = \{(\{na\}, n \cdot N^{-1}) \in \mathbb{R}^2 : 0 \leq n \leq N - 1 \} \subset [0,1)^2,$$

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where \( \{ \beta \} \) stands for the fractional part of the real number \( \beta \) and \( \alpha \) is an irrational number whose continued fraction has bounded partial quotients. Roth's theorem states that given any two-coloring \( f: \mathcal{P}_\alpha \to \{-1,+1\} \) of the set \( \mathcal{P}_\alpha \), one can find an aligned rectangle \( B \) with deviation
\[
\left| \sum_{p \in \mathcal{P}_\alpha \cap B} f(p) \right| > c(\alpha) \log N.
\]
The set \( \mathcal{P}_\alpha \) is well studied in irregularities of distribution and belongs to the class of “most uniformly” distributed \( N \)-element sets relative to aligned rectangles. What is more, Roth's proof is based on the so-called “Roth-Halász orthogonal function method” in irregularities of distribution.

The second problem is as follows: For what set of \( N \) points on the unit sphere is the sum of all \( \binom{N}{2} \) euclidean distances between points maximal, and what is the maximum?

Let \( S^k \) denote the surface of the unit sphere in \( \mathbb{R}^{k+1} \). Let \( \mathcal{P} = \{p_1, \ldots, p_N\} \) be a distribution of \( N \) points on \( S^k \). Let \( |p_i - p_j| \) denote the usual euclidean distance of \( p_i \) and \( p_j \). We define
\[
L(N, k, \mathcal{P}) = \sum_{1 \leq i < j \leq N} |p_i - p_j| \quad \text{and} \quad L(N, k) = \max_{\mathcal{P}} L(N, k, \mathcal{P}),
\]
where the maximum is taken over all \( \mathcal{P} \subset S^k \), \( \#\mathcal{P} = N \).

The determination of \( L(N, k) \) is a long-standing open problem in discrete geometry. For \( k = 1 \), the solution is given by the regular \( N \)-gon (see Fejes Tóth [13]). It is also known that for \( N = k + 2 \), the regular simplex is optimal. For \( N > k + 2 \) and \( k \geq 2 \), the exact value of \( L(N, k) \) is unknown. The reason for this is that if \( N \) is sufficiently large compared to \( k \), then there are no “regular” configurations on the sphere, so the extremal point system(s) is (are), as expected, quite complicated and “ad hoc.”

Since the determination of \( L(N, k) \) seems to be hopeless, it is natural to compare the discrete sum \( L(N, k, \mathcal{P}) \) with the integral (the solution of the “continuous relaxation” of the distance problem)
\[
\frac{N^2}{2} \cdot \frac{1}{\sigma(S^k)} \int_{S^k} |x_0 - x| \, d\sigma(x) = c_0(k)N^2, \tag{1}
\]
where \( \sigma \) denotes the surface area, \( d\sigma(x) \) represents an element of the surface area on \( S^k \), \( x_0 = (1, 0, \ldots, 0) \in \mathbb{R}^{k+1} \). Note that the constants \( c_0(k) \) can be calculated explicitly (e.g., \( c_0(1) = 2/\pi \), \( c_0(2) = 2/3 \)), and on the left-hand side of (1) the correct coefficient is \( N^2/2 \) rather than \( \binom{N}{2} \), since in the definition of \( L(N, k, \mathcal{P}) \) we can write \( 1 \leq i \leq j \leq N \) in place of \( 1 \leq i < j \leq N \) without changing the value. In this way Stolarsky [25] has discovered a beautiful identity. It states, roughly speaking, that the discrete sum \( L(N, k, \mathcal{P}) \) plus a measure of how far the set \( \mathcal{P} \) deviates from uniform distribution is constant. Thus the sum of distances is maximized by a well-distributed set of points. Combining
Stolarsky’s identity with a result in irregularities of distribution, one can obtain further information on the order of magnitude of $L(N, k)$ (see [5]).

1. Measure-theoretic discrepancy. It is time to give a brief survey of irregularities of distribution. It was initiated by a conjecture of van der Corput and the work of van Aardenne–Ehrenfest, and owes its current prominence to the contribution of K. F. Roth and W. M. Schmidt. We refer the reader to Schmidt’s book [22]; see also the forthcoming book by Chen and Beck [9].

Let $P = \{p_1, p_2, p_3, \ldots \}$ be a completely arbitrary infinite discrete set of points in euclidean $k$-space $\mathbb{R}^k$. (We can assume that $P$ has density 1, otherwise the results below are trivial.) Let $B(c, r) \subset \mathbb{R}^k$ be the ball with center $c$ and radius $r$. In 1969 Schmidt [20] proved the following pioneering result: Let $x > 1$. Then there exists a ball $B(c, r) \subset \mathbb{R}^k$ with $r \leq x$ and

$$\left| \sum_{p \in P \cap B(c, r)} 1 - \text{vol}(B(c, r)) \right| > x^{(k-1)/2 - \varepsilon}.$$  

Here the exponent $((k - 1)/2 - \varepsilon)$ of $x$ cannot be replaced by $((k - 1)/2 + \varepsilon)$. Essentially improving on the earlier result of Schmidt, the following good localization of the ball was proved [6]: Let $x > 1$. Then there exists a ball $B(c, r) \subset [0, x]^k$ such that

$$\left| \sum_{p \in P \cap B(c, r)} 1 - \text{vol}(B(c, r)) \right| > x^{(k-1)/2 - \varepsilon}.$$  

We mention next a far-reaching generalization of the case of balls. Given a compact and convex body $A \subset \mathbb{R}^k$, denote by $\sigma(\partial A)$ the surface area of the boundary $\partial A$ of $A$. The following result shows that for convex bodies the “rotation discrepancy” is always large and behaves like the square-root of the surface area of the boundary (see [6]; see also Montgomery [16]): Let $A \subset \mathbb{R}^k$ be a compact and convex body. Then there exists $A' = A(r', v', \lambda')$ obtained from $A$ by a similarity transformation of rotation $r' \in \text{SO}(k)$, translation $v' \in \mathbb{R}^k$, and contraction $\lambda' \in (0, 1]$ such that

$$\left| \sum_{p \in P \cap A'} 1 - \text{vol}(A') \right| > c(k)(\sigma(\partial A))^{1/2}. \quad (2)$$

Note that (2) is essentially the best possible (see [4]).

The next result answers an old question of Roth (see [3]): Let $P$ be an arbitrary finite set in the disc $B(0, r) \subset \mathbb{R}^2$ of radius $r$. There exists a disc-segment $A$ (i.e., an intersection of $B(0, r)$ with a half-plane) such that

$$\left| \sum_{p \in P \cap A} 1 - \text{area}(A) \right| > c \cdot r^{1/2} \cdot (\log r)^{-7/2}. \quad (3)$$
Inequality (3) is also nearly sharp. If rotation is forbidden, the situation undergoes a complete change. To avoid the considerable technical difficulties caused by higher dimensions, we restrict ourselves to the two-dimensional case. Again, let $P = \{p_1, p_2, p_3, \ldots\}$ be an infinite discrete set in $\mathbb{R}^2$. Given a compact and convex region $A \subset \mathbb{R}^2$, write

$$D(P; A) = \sum_{p \in P \cap A} 1 - \text{area}(A).$$

For any real number $\lambda \in [-1, 1]$ and any vector $v \in \mathbb{R}^2$, set $A(v, \lambda) = \{\lambda x + v : x \in A\}$. Clearly $A(v, \lambda)$ is a homothetic image of $A$. (Note that reflection across the origin is allowed, as $-1 \leq \lambda \leq 1$.) Let

$$\Delta(P; A) = \sup_{|\lambda| \leq 1, v} |D(P; A(v, \lambda))|$$

and define the (usual) discrepancy of $A$ by

$$\Delta[A] = \inf_P \Delta(P; A),$$

where the infimum is taken over all infinite discrete sets $P \subset \mathbb{R}^2$. In contrast to the "rotation discrepancy," the (usual) discrepancy $\Delta[A]$ of a convex region $A$ depends mainly on the "smoothness" of its boundary arc (see [7]). If $A$ is sufficiently smooth, then $\Delta[A]$ has essentially the same order of magnitude as for circular discs. If we have no assumption on the smoothness of $A$, we can guarantee only a much smaller discrepancy:

$$\Delta[A] > c (\log \text{area}(A))^{1/2}. \quad (4)$$

Inequality (4) probably remains true if we replace the exponent $1/2$ by the exponent $1$. If true, this is best possible. The important particular case of squares was proved by Halász [14]. Note that Halász’s theorem implies Schmidt’s solution [21] of the classical van der Corput’s conjecture. If $A$ is a polygon of "few" sides, then (4) is not very far from the truth in the sense that we cannot expect larger discrepancy than a power of $(\log \text{area}(A))$.

To get further information of the intermediate cases, one can introduce the concept of an approximability number $\xi(A)$ (which describes how well a convex region $A$ can be approximated by an inscribed polygon of few sides. The related results in [7] can be summarized as follows:

$$(\xi(A) + \log(\text{area}(A)))^{c_1} > \Delta[A] > (\xi(A) + \log(\text{area}(A)))^{c_2}.$$  

Note that if we know the equation of the boundary arc of $A$, then the determination, or at least the estimation, of $\xi(A)$ is an easy elementary problem.

We now consider very briefly the case when both rotation and contraction are forbidden, i.e., we study the supremum of the discrepancy over the family $\{A + v : v \in \mathbb{R}^2\}$. In contrast to the previous cases, the discrepancy function does not necessarily tend to infinity as $\text{area}(A)$ tends to infinity. (Let, e.g., $P = \mathbb{Z}^2$ and $A = [0, n]^2$, $n \geq 1$ integer.) However, in the case of circular discs, we can guarantee "large" discrepancy for any single value of the radius (see [8]—the problem is due to P. Erdős).

Finally, we mention the "Great Open Problem" of this field.
**CONJECTURE.** Let \( P \) be an arbitrary finite set in the cube \([0, x]^k\), \( x \geq 2\), \( k \geq 3 \). Does there exist an aligned box \( B \subset [0, x]^k \) such that

\[
\left| \sum_{p \in P \cap B} 1 - \text{vol}(B) \right| > c(k)(\log x)^{k-1}?
\]

Since 1954 the best lower bound is \((\log x)^{(k-1)/2}\) (see Roth [17]).

The proofs of all these lower bounds are based on tools in harmonic analysis (e.g., modified Rademacher functions, Riesz products, summability kernels). The common idea of all different approaches—to "blow up the trivial error." Note that the use of Fourier analysis in the opposite direction (i.e., to show the uniformity of sequences) is a classical idea and goes back to H. Weyl ("Weyl's criterion" and its quantitative versions, e.g., "Erdős-Turán inequality").

The proofs of the upper bounds are based on ideas from number theory, probability theory, and "combinatorial discrepancy theory."

2. **Combinatorial discrepancy.** The basic problem of the so-called "combinatorial discrepancy theory" is how to color with two colors a set as uniformly as possible with respect to a given family of subsets. What we want to achieve is that the coloring be nearly balanced in each of the subsets considered. As a beautiful example, we mention Roth's theorem on long arithmetic progressions [18]. Roth proved that coloring the integers from 1 to \( N \) red and blue in any fashion, there exists an arithmetic progression such that the difference of the numbers of red and blue terms in this progression has absolute value \( > c \cdot N^{1/4} \).

In this section we discuss some general upper bounds concerning hypergraphs. They are interesting on their own, but also they have applications to different structures.

Let \( X \) be an arbitrary finite set and \( \mathcal{H} = \{Y_1, Y_2, Y_3, \ldots\} \) an arbitrary family of subsets of \( X \). We would like to find a two-coloring \( f: X \to \{-1, +1\} \) of the underlying set \( X \) such that \( \max_{Y \in \mathcal{H}} |\sum_{x \in Y} f(x)| \) is as small as possible. In other words, let

\[
\text{dis}(\mathcal{H}) = \min_{f} \max_{Y \in \mathcal{H}} \left| \sum_{x \in Y} f(x) \right|,
\]

where the minimum is taken over all \( f: X \to \{-1, +1\} \). We call \( \text{dis}(\mathcal{H}) \) the combinatorial discrepancy of the family \( \mathcal{H} \).

Let \( d(\mathcal{H}) \) be the maximum degree of \( \mathcal{H} \), i.e.,

\[
d(\mathcal{H}) = \max_{x \in X} \#\{Y \in \mathcal{H}: x \in Y\}.
\]

The following result gives an upper bound on \( \text{dis}(\mathcal{H}) \) which depends only on
\(d(\mathcal{H})\), i.e., the “local size” of \(\mathcal{H}\) (see Fiala and Beck [10]): For any finite family \(\mathcal{H}\),
\[
\text{dis}(\mathcal{H}) < 2d(\mathcal{H}).
\] (5)

Note that the proof of (5) gives a good (polynomial time) algorithm which constructs the two-coloring. There is an important point to emphasize here: In many applications \(d(\mathcal{H})\) is much less than \(#X\) and \(#\mathcal{H}\). As an illustration, we derive the upper bound \(T(N) < c(\log N)^4\) (see Introduction), which is clearly equivalent to the following result: Let \(A = (a_{ij})\), where \(a_{ij} = 0\) or \(1\), be a matrix of size \(N \times N\). Then there exist “signs” \(\varepsilon_{ij} = \pm 1\) such that
\[
\left| \sum_{i=1}^{s} \sum_{j=1}^{t} \varepsilon_{ij} a_{ij} \right| < c(\log N)^4 \quad \text{for all } s, t \in \{1, 2, \ldots, N\}.
\] (6)

We can assume that \(N = 2^l\), where \(l\) is an integer. For \(0 \leq p, q \leq l\), we partition the matrix \(A\) into \(2^{p+q}\) submatrices, splitting the horizontal side of the matrix into \(2^p\) equal pieces and the vertical side of the matrix into \(2^q\) equal pieces. There are \((l+1)^2 \sim (\log N)^2\) such partitions. Let us call a submatrix of \(A\) special if it occurs in one of these partitions, and let \(\mathcal{H}\) be the collection of all these special submatrices. Then by (5), there exists an assignment of \(\pm 1\)'s so that the absolute value of the sum of signed entries in each of the special submatrices is less than 
\[
2d(\mathcal{H}) < 2(l+1)^2.
\] Note, however, that any submatrix of \(A\) containing the lower left corner of \(A\) is the union of at most \(l^2\) disjoint special submatrices, and (6) follows.

In higher dimensions, the same argument gives the following generalization of (6): Let \(A = (a_n)\), where \(a_n = 0\) or \(1\), be a \(k\)-dimensional matrix of size \(N \times \cdots \times N\). Then there exist “signs” \(\varepsilon_n = \pm 1\) such that
\[
\left| \sum_{n : n \leq m} \varepsilon_n a_{n} \right| < c(k)(\log N)^{2k}
\] (7)
for all \(m = (m_1, \ldots, m_k)\) satisfying \(1 \leq m_i \leq N\) (\(1 \leq i \leq k\)). Here \(n \leq m\) if and only if \(n_i \leq m_i\) for all \(i \in [1, k]\).

We conjecture that inequality (5) can be improved to 
\[
\text{dis}(\mathcal{H}) < (d(\mathcal{H}))^{1/2 + \varepsilon},
\] where \(d = d(\mathcal{H}) > d_0(\varepsilon)\). The following result justifies this conjecture when both \(#X\) and \(#\mathcal{H}\) are “subexponential” functions of \(d = d(\mathcal{H})\) ([7]—see inequality (9) below).

Let \(X\) be a finite set and \(\mathcal{H}\) a family of subsets of \(X\). Suppose that there is a second family \(\mathcal{G}\) of subsets of \(X\) such that
\begin{enumerate}
  \item \(d(\mathcal{G}) \leq d\), and
  \item every \(Y \in \mathcal{H}\) can be represented as the union of at most \(t\) disjoint elements of \(\mathcal{G}\).
\end{enumerate}
Then
\[
\text{dis}(\mathcal{H}) < c(t \cdot d \cdot \log d \cdot \log \#\mathcal{H})^{1/2} \cdot (\log \#X).
\] (8)
In the particular case $\mathcal{G} = \mathcal{H}$, we obtain

$$\text{dis}(\mathcal{H}) < c(d(\mathcal{G}))^{1/2} \cdot \log \#X \cdot \log \#\mathcal{H}.$$  \hfill (9)

We next apply (8) to improve on (7). Let $X = \{ n: a_n = 1 \}$, let $\mathcal{G}$ be the family of all submatrices $(a_n)$ ($n \leq m$), $\mathcal{G}$ be the family of all $k$-dimensional special submatrices, $N = 2^l$, $d = (l + 1)^k$, and $t = l^k$. By (8) we have that

$$\sum_{n \leq m} |< n, a_n : n < m >| \leq c(k, \varepsilon) (\log N)^{k+3/2+\varepsilon}$$

for all $m$.

The proof of (8) is "nonconstructive." The new idea is a combination of probabilistic arguments with the pigeon-hole principle. As far as we know, the first application of this idea is in [2]. (It was shown that Roth's theorem on long arithmetic progressions is essentially the best possible.) Later the same method was utilized by Spencer and Beck (see [11, 12, 24]).

For further results in combinatorial discrepancy theory, see Vera Sós [23] and Lovász, Spencer, Vesztergombi [15].

Finally, we have to remark that there are no general lower bounds on the combinatorial discrepancy of hypergraphs. To illustrate the difficulties, we mention a more than fifty-year-old question of Erdős.

**Conjecture (Erdős).** Let $f(n) = \pm 1$ be a function on the set of positive integers. Given arbitrary large constant $c$ there is a $d$ and an $m$ so that $|\sum_{i=1}^{m} f(i \cdot d)| > c$.

**References**

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