Multidimensional Hypergeometric Functions in Conformal Field Theory, Algebraic K-Theory, Algebraic Geometry

Alexandre Varchenko

Moscow Institute of Gas and Oil, Leninski Prospekt 65, 117917 Moscow, USSR

Rudolf Arnheim in the book *Visual Thinking* (L.A. 1969) writes that usually concepts tend to crystallize into simple, well-shaped forms. They are tempted by Platonic rigidity. This creates troubles when the range they are intended to cover includes relevant qualitative differences. The variations can be so different from each other that to see them as belonging to one family of phenomena requires mature understanding. To the young mind, they look as different from each other as did the morning star from the evening star to the ancients.

The notion of a general hypergeometric function was introduced by I.M. Gelfand in the mid 80s. Now it is clear that general hypergeometric functions play a major role in interesting parts of mathematics such as Conformal Field Theory, Representation Theory, Algebraic K-Theory, Algebraic Geometry and provide new connections among them.

The general hypergeometric functions are generalizations of the Euler beta-function. The beta-function is the integral of a product of powers of linear functions over the segment. In the generalization the segment is replaced by a polytope and the integral

$$I(\Delta, \{f_i\}, \{\alpha_j\}) = \int_{\Delta} f_1^{\alpha_1} \ldots f_n^{\alpha_n} \, dx_1 \ldots dx_n$$

is considered as a function of the polytope $\Delta \subset \mathbb{R}^n$, the linear functions $\{f_i\}$, and the exponents $\{\alpha_j\} \subset \mathbb{C}$. The simplest examples are the classical hypergeometric function, the Euler dilogarithm, the volume of a polytope. The systematic study of the general hypergeometric functions was begun only recently in works of I.M. Gelfand's school and K. Aomoto.

There are three basic reasons for the appearance of general hypergeometric functions: the general hypergeometric functions satisfy remarkable differential equations, the general hypergeometric functions satisfy remarkable functional equations, the general hypergeometric functions, as analytic functions of their arguments, have remarkable monodromy groups.
1. Functional Equations

1.1 Volume of Polytope

The volume of a convex polytope in $\mathbb{R}^3$ has the following properties. The volume does not change under movements of the polytope. If a polytope is divided into two parts by a plane, then the volume of the polytope is equal to the sum of the volumes of the parts. These properties are functional equations of the volume considered as a function on the space of all convex polytopes. The properties of the volume suggest the following definition. The group of polytopes in $\mathbb{R}^3$ is the abelian group generated by the symbols $(A)$, where $A \subset \mathbb{R}^3$ is any convex polytope, subject to the relations

(a) $(A) = (gA)$ for any motion $g$.
(b) $(A) = (A_1) + (A_2)$, if $A$ is divided by a plane into parts $A_1, A_2$.

Is an element of this group uniquely determined by its volume? Is a regular tetrahedron equivalent to a cube of the same volume? These questions form the content of the third Hilbert problem. The third Hilbert problem was solved in Dehn’s articles in 1900–1902, before it was published. It turned out that the regular tetrahedron cannot be composed from a cube because the two have different Dehn invariants.

Consider an edge of a polytope. An edge has two characteristics: the length $l$ and the angle $\theta$ at the edge. The angle is defined up to a multiple of $\pi$ and lies in $\mathbb{R}/\pi\mathbb{Z}$. The Dehn invariant of a polytope is the expression $D(A) = \sum l_i \otimes \theta_i$, where the sum is taken over all edges of the polytope. The Dehn invariant is an element of the group $\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/\pi\mathbb{Z})$. It is obvious that the invariant does not change under movements of the polytope and is additive under cuttings. It is easy to see that the Dehn invariant of a cube is equal to zero (as is that of any prism) and that the Dehn invariant of a regular tetrahedron is not equal to zero. So a regular tetrahedron cannot be composed from a cube. According to Sydler, 1965, the equivalence class of a polytope is uniquely determined by its volume and Dehn invariant, see also [C1, Du, DuPS, DuS, S].

This example demonstrates the scheme leading from a hypergeometric function to interesting algebraic concepts. Given a function on some space satisfying some functional equations one considers the abelian group generated by the points of the space subject to relations given by the functional equations of the initial function. If the initial function satisfies some differential equations then the group has additional structure.

1.2 Example

The Euler dilogarithm is the function defined by the power series

$$\text{Li}_2(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^2}$$

for $|z| < 1$.

The dilogarithm is a hypergeometric function. It has the integral representation

$$\text{Li}_2(z) = \int_{\Delta} \frac{dx}{x} \wedge \frac{dy}{y}$$
where the triangle $\Delta$ is shown in Fig. 1. The dilogarithm satisfies the differential equation
\[ d \ln(1-t) = \ln(1-t) \, dt \]
and the functional equation
\[
\begin{align*}
\text{Li}_2(x) - \text{Li}_2(y) + \text{Li}_2(y/x) - \text{Li}_2\left(\frac{y(1-x)}{x(1-y)}\right) & + \text{Li}_2\left(\frac{1-x}{1-y}\right) \\
& = \frac{n^2}{6} - \ln(x) \ln(1-y).
\end{align*}
\]
for $0 < y < x < 1$.

The algebraic construction. The Bloch group $B_2$ of a field $F$ is the abelian group generated by the symbols $(t)$, where $t \in F \setminus \{0, 1\}$, subject to the relations
\[
(x) - (y) + (y/x) - \left(\frac{y(1-x)}{x(1-y)}\right) + \left(\frac{1-x}{1-y}\right) = 0
\]
for any $x, y \in F \setminus \{0, 1\}$.

Consider the multiplicative group $F^*$ of a field $F$ and its exterior square $F^* \wedge F^*$. The map
\[
t \mapsto (t) \wedge (1-t),
\]
sending the symbol $(t), t \in F \setminus \{0, 1\}$, to the element $(t) \wedge (1-t)$ of the group $F^* \wedge F^*$, has a remarkable property. It sends the alternating sum of the elements on the left-hand side of (1) to the zero element of the group $F^* \wedge F^*$. This gives a well-defined homomorphism
\[
S : B_2 \to F^* \wedge F^*,
\]
called the Bloch complex.

The homology groups of the Bloch complex are connected with the Quillen $K$-groups of the field $F$. In algebraic $K$-theory with any field $F$ there is associated the sequence of groups $K_n(F), n \geq 0$. For these groups a multiplication $K_p \otimes K_q \to K_{p+q}$ is defined.

**Theorem** (Matsumoto, Suslin).

1) $\text{Coker } S \simeq K_2(F)$. 

\[
\text{Fig. 1}
\]
2) \((\text{Ker} \ S) \otimes \mathbb{Q} \cong K_3^{\text{ind}}(F) \otimes \mathbb{Q}\), where \(K_3^{\text{ind}} = K_3/K_1^3\) is the indecomposable part of \(K_3\).

This theorem gives an elementary definition of \(K_2\) and the indecomposable part of \(K_3\) (modulo torsion). An interesting problem is to find elementary definitions of all the groups \(K_n(F)\). Conjecturally such definitions arise from functional and differential equations of polylogarithmic functions.

There are several possible generalizations of the logarithm and the Euler dilogarithm \([A3, \ Le, \ GM, \ HaM]\). One of them is the Aomoto polylogarithms.

1.3 Aomoto Polylogarithms

A simplex in \(P^n(\mathbb{C})\) is an ordered set \(L = (L_0, L_1, \ldots, L_n)\) of hyperplanes. A simplex defines the differential \(n\)-form
\[
\omega_L = d \ln(z_1/z_0) \wedge \cdots \wedge d \ln(z_n/z_0),
\]
where \(z_i = 0\) is a homogeneous equation of \(L_i\). With a second simplex \(M = (M_0, \ldots, M_n)\) \(n\)-chain \(\Delta_M\) in \(P^n(\mathbb{C}) \setminus L\) is connected. \(\Delta_M\) is a curved oriented

\[\begin{align*}
P^2(\mathbb{C}) & \quad \text{Fig. 2} \\
L_0 & \quad L_1 \\
L_0 & \quad L_2
\end{align*}\]

\(n\)-simplex with boundary in \(M\), see Fig. 2. With a pair of simplices an integral
\[
a_n(L; M) = \int_{\Delta_M} \omega_L.
\]
is associated called the Aomoto polylogarithm of order \(n\). The integral depends on the choice of the chain \(\Delta_M\), but does not change under its deformation. The Aomoto polylogarithm has the following properties.

(2) Antisymmetry. The integral is antisymmetric with respect to renumbering of the hyperplanes of the first or the second simplices.

(3) Additivity with respect to a form. If \(L_0, \ldots, L_{n+1}\) are \((n + 2)\) hyperplanes and \(L^i := (L_0, \ldots, \hat{L}_i, \ldots, L_{n+1})\), then
\[
\sum_{j=0}^{n+1} (-1)^j a_n(L^j; M) = 0.
\]

(4) Additivity with respect to a chain. If \(M_0, \ldots, M_{n+1}\) are \((n + 2)\) hyperplanes, then
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\[ \sum_{j=0}^{n+1} (-1)^j a_n(L; M^j) = 0 \]

for a suitable choice of \( A_{M^j} \).

(5) Projective invariance. For every \( g \in PGL(n+1, \mathbb{C}) \) if \( A_{gM} = gA_M \), then

\[ a_n(gL; gM) = a_n(L; M). \]

For example, an Aomoto polylogarithm of order 1 is defined by two pairs of points \( L = (L_0, L_1) \), \( M = (M_0, M_1) \) on \( P^1(\mathbb{C}) \) and is equal to an integral of the form \( \omega_L = d \ln(z_1/z_0) \) over a path going from \( M_1 \) to \( M_0 \). \( a_1(L; M) \) is equal to the logarithm of the cross ratio of the four points \( (L_0, L_1, M_0, M_1) \). The Euler dilogarithm is a special case of the Aomoto dilogarithm, see Fig. 1.

There are two connections of polylogarithms of different orders: the multiplication and the differential equation.

(6) The product of Aomoto polylogarithms of orders \( p, q \) may be expressed as a sum of Aomoto polylogarithms of order \( p + q \), see [BGSV].

(7) The differential of the polylogarithm of order \( n \), considered as a function on the space of all configurations \( (L, M) \) may be expressed through suitable polylogarithms of orders \( n - 1 \) and 1 [A3].

Aomoto polylogarithms describe parameters of the mixed Hodge structure of cohomology groups of a pair \( P^n(\mathbb{C}) \setminus L, M \setminus M \cap L \), where \( L, M \) are configurations of hyperplanes, see [BMS, BGSV]. For example if two points \( L_0, L_1 \in P^1(\mathbb{C}) \) are removed and two points \( M_0, M_1 \in P^1(\mathbb{C}) \) are identified then the mixed Hodge structure of the first cohomology group of this space is defined by the number \( \exp(a_1(L_0, L_1; M_0, M_1)) \).

1.4 Hopf Algebra of Pairs of Simplices

Taking properties of Aomoto polylogarithms as a starting point, it is possible to suggest a definition of a graded Hopf algebra \( A(F) = A_0 \oplus A_1 \oplus A_2 \oplus \cdots \) for any field \( F \) [BMS, BGSV]. Here \( A_0 := \mathbb{Z} \), \( A_n \) is the abelian group, generated by the symbols \( (L; M) \), where \( L = (L_0, \ldots, L_n) \), \( M = (M_0, \ldots, M_n) \) are ordered sets of hyperplanes in \( P^n(F) \), subject to relations similar to (2)–(5). The multiplication \( \mu_{n,q} : A_p \otimes A_q \to A_{p+q} \) and the comultiplication \( v_n = \bigoplus_{p=0}^{n} v_{p,n-p} : A_n \to \bigoplus_{p=0}^{n} A_p \otimes A_{n-p} \) of the Hopf algebra are modeled by properties (6), (7).

There is the sequence of complexes \( A[n] \), \( n \in \mathbb{Z}_+ \), associated with any graded Hopf algebra \( A \). The simplest of these complexes are

\[ A[1] : 0 \to A_1 \to 0, \]
\[ A[2] : 0 \to A_2 \overset{\mu_{1,1}}{\to} A_1 \otimes A_1 \to 0, \]
\[ A[3] : 0 \to A_3 \overset{\mu_{2,1} \otimes v_{1,1}}{\to} A_2 \otimes A_1 \oplus A_1 \otimes A_2 \overset{v_{1,1} \otimes 1 - 1 \otimes v_{1,1}}{\to} A_1 \otimes A_1 \otimes A_1 \to 0. \]

The complex \( A[n] \) is concentrated in degrees from 1 to \( n \). Conjecturally, the cohomology groups of these complexes give the K-groups of the field \( F \) modulo torsion.
Conjecture (Beilinson) [BGSV].

\[ K_n(F) \otimes \mathbb{Q} \simeq \bigoplus_{j=0}^{[(n-1)/2]} H^{n-2j}(A[n-j] \otimes \mathbb{Q}). \]

Theorem [BGSV]. The conjecture holds for \( n \leq 3 \):

\[ K_1 \simeq H^1(A[1]), \quad K_2 \simeq H^2(A[2]), \quad K_3, \mathbb{Q} \simeq H^3(A[3], \mathbb{Q}) \oplus H^1(A[2], \mathbb{Q}). \]

In the last years efforts have been undertaken to construct a motivic cohomology theory of algebraic manifolds which would be an arithmetical variant of the singular cohomology theory. According to Beilinson a category of motivic sheaves over the spectrum of a field \( F \) is a category of graded modules over a suitable algebra and a possible candidate for it could be the algebra \((A(F) \otimes \mathbb{Q})^*\) dual to the algebra \( A(F) \otimes \mathbb{Q} \).

1.5 Bloch-Wigner Function

The Euler dilogarithm may be continued to a multivalued analytic function on \( \mathbb{C} \setminus \{0, 1\} \). The Bloch-Wigner function is its imaginary part

\[ D(z) = \text{Im}(\text{Li}_2(z)) + \text{arg}(1 - z) \ln|z|, \]

see [B2, Z]. The Bloch-Wigner function has the following properties.

(8) \( D(z) \) is single-valued real analytic on \( \mathbb{C} \) except at the points 0 and 1, where it is only continuous.

(9) \[ D(x) - D(y) + D(y/x) - D(1 - x) + D \left( \frac{y(1 - x)}{x(1 - y)} \right) = 0 \]

for any \( x, y \in \mathbb{C} \setminus \{0, 1\} \).

In particular, for any field \( F \subset \mathbb{C} \) the Bloch-Wigner function defines a homomorphism \( B_2(F) \to \mathbb{R} \) of the Bloch group to real numbers.

1.6 Polylogarithms and Zeta-Function

Let \( F \) be an algebraic number field of degree \( n \) over \( \mathbb{Q} \) with \( r_1 \) real and \( r_2 \) complex places, \( r_1 + 2r_2 = n \). Let \( \zeta_F(s) \) be the Dedekind zeta-function of \( F \). The value \( \zeta_F(2) \) is expressed in terms of the values of the Bloch-Wigner function at points of the field \( F \).

Theorem [Bo, B, Su, Z]. In the Bloch complex of \( F \) the group \( \text{Ker} \, S \) is isomorphic (modulo torsion) to \( \mathbb{Z}^{r_2} \). The co-volume of the image of the map \( \text{Ker} \, S \to \mathbb{R}^{r_2} \), defined by the composition of complex imbeddings and the Bloch-Wigner function, is a non-zero rational multiple of \( \pi^{2(r_1 + r_2)} |d_F|^{1/2} \zeta_F(2) \), where \( d_F \) is the discriminant of \( F \).

D. Zagier conjectured that for any natural number \( m \) the value \( \zeta_F(m) \) has similar expression in terms of values of the classical polylogarithms of order at most \( m \) at points of \( F \), [Z]. Recently A. Goncharov proved this conjecture for \( m = 3 \).
Let $\mathbb{Z}[F \setminus \{0, 1\}]$ be the free abelian group generated by the symbols $(t), t \in F \setminus \{0, 1\}$. $\mathbb{Q}[F \setminus \{0, 1\}] := \mathbb{Z}[F \setminus \{0, 1\}] \otimes \mathbb{Q}$. For any function $D : \mathbb{C} \setminus \{0, 1\} \to \mathbb{R}$ there is a homomorphism $D : \mathbb{Q}[\mathbb{C} \setminus \{0, 1\}] \to \mathbb{R}, \ D : \sum n_i(t_i) \to \sum n_iD(t_i)$.

Let $\{a_j\}, j = 1, \ldots, r_1 + 2r_2,$ be all possible imbeddings $F \to \mathbb{C}, \overline{a_{r_1+k}} = a_{r_1+r_2+k}$.

**Theorem [Go].** There exist $z_1, \ldots, z_{r_1+r_2} \in \mathbb{Q}[F \setminus \{0, 1\}]$, such that

$$\zeta_{r}(3) = \pi^{3r} \left| d_{r} \right|^{-1/2} \det(D_3(z_j(a_i))),$$

where $j = 1, \ldots, r_1 + r_2$,

$$D_3(z) = \Re(Li_3(z) - \ln|z|Li_2(z) + \frac{1}{3} \ln^2|z|Li_1(z)),$$

$L_i(z), n \geq 1$, is the classical polylogarithm of order $n$, defined by the power series

$$Li_n(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^n} \quad \text{for } |z| < 1.$$  

## 2. Hypergeometric Integrals

### 2.1 Classical Hypergeometric Function

The hypergeometric series

$$F(a, b; c; z) = 1 + \frac{ab}{1c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2c(c+1)} z^2$$

$$+ \frac{a(a+1)(a+2)b(b+1)(b+2)}{1 \cdot 2 \cdot 3c(c+1)(c+2)} z^3 + \cdots$$

satisfies the differential equation

$$z(1 - z)F'' + (c - (a + b + 1)z)F' - abF = 0$$

and has the integral representation

$$\frac{\Gamma(b)\Gamma(c - b)}{\Gamma(c)} F(a, b; c; z) = \int_{1}^{\infty} t^{a-c}(t - 1)^{c-b-1}(t - z)^{-a} \, dt.$$

Thus the classical hypergeometric function has three definitions: as a power series, as the solution of a differential equation, as an integral depending on a parameter.

These objects are associated with the family of the configurations of the triples of the points $0, 1, z$ of the complex line and are generalized naturally to the case of a family of configurations of hyperplanes in an affine space $[A, G, GGZ, GKZ]$.

### 2.2 Cohomology of Complement of Configuration of Hyperplanes

Let $\mathcal{C}$ be a finite set of hyperplanes in $\mathbb{C}^n$. Choose a linear equation $f_H = 0$ for any hyperplane. Define the closed differential form $df_H/f_H$. The Orlik-Solomon algebra is the exterior algebra generated by 1 and the forms $df_H/f_H, H \in \mathcal{C}$.  

Theorem [Ar1, Bri]. The Orlik-Solomon algebra is naturally isomorphic to the complex cohomology ring of the complement of a configuration.

In other words every cohomology class can be represented as a polynomial in the forms \(df_H/f_H\), such a polynomial defines the zero class only if it is equal to zero.

Example [Ar1]. The Orlik-Solomon algebra of the configuration of all diagonal hyperplanes \(t_i - t_j = 0\) in \(\mathbb{C}^n\) is isomorphic to the exterior algebra generated by 1 and the symbols \(w_{ij}\) subject to the relations \(w_{ij} = w_{ji}, w_{ij} \wedge w_{jk} + w_{jk} \wedge w_{ki} + w_{ki} \wedge w_{ij} = 0\) for pairwise different \(i, j, k\). \(P(t) = (1 + t)(1 + 2t)\ldots(1 + (n - 1)t)\) is the Poincaré polynomial of the algebra, see also [Or, OS].

V.I. Arnold computed this example in connection with study of superpositions of algebraic functions and Hilbert's 13th problem; see [Ar2].

A configuration is called weighted if a complex number \(a(H)\) is assigned to each hyperplane \(H\). The weights define the function

\[ l_\alpha = \prod_{H \in \mathcal{C}} f_H^{a(H)}. \]

This function is a multivalued function on the complement of a configuration. The differential of this function has the form

\[ dl_\alpha = l_\alpha \sum_{H \in \mathcal{C}} a(H) df_H/f_H. \]

A hypergeometric differential form of a weighted configuration is any form \(l_\alpha \omega\), where \(\omega\) is a differential form of the Orlik-Solomon algebra.

Hypergeometric forms form a finite-dimensional complex, as the differential of a hypergeometric form is a hypergeometric form:

\[ d(l_\alpha \omega) = l_\alpha \sum_{H \in \mathcal{C}} a(H) df_H/f_H \wedge \omega. \]

The weight local system \(\mathcal{S}(\alpha)\) on the complement of a configuration is the complex one-dimensional local system of coefficients with the monodromy around a hyperplane \(H\) equal to the multiplication by \(\exp(-2\pi i a(H))\).

The cohomology of the complement of a configuration with coefficients in the weight local system is computed by the complex of hypergeometric forms. More precisely for any number \(t\) denote by \(ta\) the weights \(H \mapsto ta(H), H \in \mathcal{C}\), homothetic to the initial ones.

Theorem [SV3]. For almost all \(t \in \mathbb{C}\) the cohomology of the finite dimensional complex of hypergeometric forms with weights \(ta\) is naturally isomorphic to the cohomology of the complement of a configuration with coefficients in \(\mathcal{S}(ta)\).

For example, this is true for \(t = 0\) according to the Arnold-Brieskorn theorem. Exceptional values of \(t\) form a discrete set. Conjecturally the exceptions form explicitly given arithmetical progressions [Ao1].
2.3 Determinant Formulas

The Euler beta-function is the alternating product of Euler gamma-functions:

\[ B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}. \]

There is a generalization of this formula to the case of a configuration of hyperplanes in an affine space \([V, \mathcal{G}, A]\).

**Example.** Consider the configuration of three points \(z_1, z_2, z_3\) on a line, see Fig. 3. The point \(z_j\) is the zero of the function \(f_j = t - z_j\). Put

\[ l_1 = (t - z_1)^{\alpha_1}(t - z_2)^{\alpha_2}(t - z_3)^{\alpha_3}, \]
\[ \omega_1 = \alpha_1 l_1 d(t - z_1)/(t - z_1), \quad \omega_2 = \alpha_2 l_1 d(t - z_2)/(t - z_2), \]

then

\[ \det \left( \int_{\Delta_i} \omega_j \right) = \frac{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)\Gamma(\alpha_3 + 1)}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + 1)} \prod_{i \neq j} f_i^{\omega_i}(z_j). \]

Thus the determinant of integrals of basic hypergeometric forms of a configuration over all bounded components of the complement of the configuration is equal to the product of values of powers of linear functions at the vertices of the configuration up to a multiplicative constant equal to an alternating product of values of the gamma function \([V]\).

The formula has an arithmetical analog. F. Loeser extended it to the case of a configuration of hyperplanes in an affine space over a finite field \([Lo]\). In this case the gamma-functions are replaced by Gauss sums, the determinant of the hypergeometric integrals is replaced by an alternating product of the determinants of the Frobenius operator in suitable cohomology groups.

3. Hypergeometric Functions and Representation Theory of Lie Algebras

The appearance of hypergeometric functions in the representation theory of Kac-Moody algebras, their quantum deformations, and in the Conformal Field Theory (CFT) is connected with integrals of the form

\[ I(t_1, \ldots, t_n) = \int \prod_{1 \leq j < k \leq N} (t_j - t_k)^{2jk} dt_{n+1} \wedge \cdots \wedge dt_N, \]

\[ N = m + n. \]
Such integrals correspond to special configurations. The characteristics of these configurations, in particular the homology groups with twisted coefficients of the complement, the complex of hypergeometric forms are interpreted as objects of the representation theory of Kac-Moody algebras. Such integrals satisfy the differential equation which is described in terms of representation theory and is known in CFT as the Knizhnik-Zamolodchikov equation. The branching of such integrals is described in terms of quantum groups corresponding to Kac-Moody algebras.

The appearance of these integrals in physical models isn't surprising. Imagine a model in which points \( t_1, \ldots, t_N \) of the line pairwise interact. The interaction is described by the function \((t_j - t_k)\lambda_{jk}\). In this case the average of the interaction over all positions of the last \( m \) points is an integral characteristic of the first \( n \) points, described by the hypergeometric integral.

In applications the constants of the interaction \( \lambda_{jk} \) have the form

\[
\lambda_{jk} = B(v_j, v_k)/\kappa,
\]

where \( v_1, \ldots, v_N \) are vectors of some complex linear space \( V \), \( B \) is a symmetric bilinear form on \( V \), and \( \kappa \) is a complex parameter of the model.

In applications to the representation theory of Kac-Moody algebras, \( V \) is the dual space to the Cartan subalgebra of the Kac-Moody algebra, \( B \) is the Killing form on it. The vectors \( v_{n+1}, \ldots, v_N \) corresponding to the averaged points belong to the set of simple negative roots. The vectors \( v_1, \ldots, v_n \) corresponding to the parameters of the integral are the highest weights of the representations of the Kac-Moody algebra.

### 3.1 Hypergeometric Construction

Assume given

(a) natural numbers \( n \leq N, N = m + n \);
(b) a complex linear space \( V \), symmetric bilinear form \( B \) on \( V \) and an ordered set of (weight) vectors \( v_1, \ldots, v_N \in V \), not necessarily different.

From these data the construction builds a complex linear space \( W \) and a differential equation on a \( W \)-valued function \( \phi(z_1, \ldots, z_n) \):

\[
d\phi = \kappa^{-1} \sum_{1 \leq j < k \leq n} \Omega_{jk} \phi d(z_j - z_k)/(z_j - z_k),
\]

where \( \Omega_{jk} : W \to W \) are suitable linear operators, \( \kappa \) is a complex parameter.

The Construction. Consider the configuration of all the diagonal hyperplanes \( t_j = t_k \) in \( \mathbb{C}^N \). Define the weights of the diagonals: \( \lambda_{jk} = B(v_j, v_k)/\kappa \).

The weights define the complex of hypergeometric forms, the one-dimensional complex local system \( \mathcal{S} \) on the complement of the diagonals with the monodromy around the diagonal \( t_j = t_k \) equal to the multiplication by \( \exp(2\pi i \lambda_{jk}) \). Hypergeometric forms have well-defined integrals over chains with coefficients in \( \mathcal{S} \).

Let \( \mathbb{C}^N \to \mathbb{C}^n \) be the projection on the first \( n \) coordinates. A fiber over a point \( z \) is the space \( \mathbb{C}^m \) in which the diagonals cut a configuration depending on \( z \). Denote it by \( \mathcal{C}(z) \).
A point of the base is called *discriminantal* if the configuration in the fiber over it is degenerate. The discriminantal points form the discriminant, the configuration of all diagonal hyperplanes. Over the complement of the discriminant, fibers with distinguished configurations form a locally trivial bundle. The fundamental group of the complement of the discriminant is the pure braid group on \( n \) strings.

Consider the restriction \( \mathcal{S}(z) \) to the fiber \( \mathbb{C}^m \backslash \mathcal{S}(z) \) over a point \( z \) of the local system \( \mathcal{S} \), and the top homology group \( H_m(\mathbb{C}^m \backslash \mathcal{S}(z), \mathcal{S}(z)) \) of the fiber with coefficients in this restriction. The *homology bundle* is the complex vector bundle over the complement of the discriminant with fiber \( H_m(\mathbb{C}^m \backslash \mathcal{S}(z), \mathcal{S}(z)) \) over the point \( z \).

The homology group \( H_m(\mathbb{C}^m \backslash \mathcal{S}(z), \mathcal{S}(z)) \) depends on \( z \) and is uniquely translated along paths in the base. The *Gauss-Manin connection* is this integrable connection on the homology bundle.

The *monodromy representation* of the Gauss-Manin connection is the representation of the pure braid group in the automorphism group of the homology group, induced by translations of the homology group along loops in the complement of the discriminant.

The monodromy representation gives the well-known Burau representation of the braid group in the special case of one-dimensional fiber and equal weights [K2, GiS, Gi]. As will be explained the representations of the braid groups appearing in the theory of quantum groups are closely connected with the monodromy representations of the constructed Gauss-Manin connections.

An integral of a hypergeometric form on \( \mathbb{C}^N \) over any cycle in a fiber is equal to zero certainly if this form is the differential of some form or the restriction of this form on any fiber equals zero.

Accordingly we define the *hypergeometric cohomology group* as

\[
H^m = \Omega^m \cap (Z^m + d\Omega^{m-1}),
\]

where \( \Omega^m \) is the space of hypergeometric \( m \)-forms on \( \mathbb{C}^N \), \( Z^m \subset \Omega^m \) is the subspace of forms with zero restriction on any fiber, \( d\Omega^{m-1} \subset \Omega^m \) is the subspace of differentials of hypergeometric \( (m-1) \)-forms.

The integration defines the homomorphism

\[
i : H_m(\mathbb{C}^m \backslash \mathcal{S}(z), \mathcal{S}(z)) \to (H^m)^*
\]
of the homology group over a point \( z \) to the space independent of \( z \). The homomorphism depends on \( z \) and the parameter \( \kappa, \iota = \iota(z, \kappa) \).

**Theorem** [SV]. For almost all \( \kappa^{-1} \) with exceptions in a suitable discrete subset of \( \mathbb{C} \) the homomorphism \( \iota(z, \kappa) \) is an isomorphism for all points \( z \) outside the discriminant.

**Theorem** [Ao, SV]. There exist linear operators \( \Omega_{jk} : (\mathcal{H}^m)^* \to (\mathcal{H}^m)^*, 1 \leq j < k \leq n \), with the following properties. For any locally constant homology class \( \Delta(z) \in H_n(\mathbb{C}^m \setminus \mathcal{O}(z), \mathcal{S}(z)) \) the \((\mathcal{H}^m)^*\)-valued function \( \phi(z) := \iota(z, \kappa)[\Delta(z)] \) satisfies the differential equation

\[
d\phi = \kappa^{-1} \sum_{1 \leq j < k \leq n} \Omega_{jk} \phi d(z_j - z_k)/(z_j - z_k).
\]

(10)

In other words hypergeometric integrals satisfy the differential equation (10) and for a general \( \kappa \) all solutions of the equation are given by the integrals.

This picture has a symmetry group. Any permutation of the coordinates in \( \mathbb{C}^N \) accordingly enumerates the weight vectors \( v_1, \ldots, v_N \in V \). Any permutation of the last \( m \) coordinates preserving the ordered set of the weight vectors preserves the fibers of the projection, the weights of the diagonals, acts on the complex of hypergeometric forms, on homology groups of fibers, on solutions of the differential equation (10).

The twisted homology group \( H_n(\mathbb{C}^m \setminus \mathcal{O}(z), \mathcal{S}(z))_{\text{ant}} \) is the antisymmetric part of the group \( H_n(\mathbb{C}^m \setminus \mathcal{O}(z), \mathcal{S}(z)) \) with respect to the action of the permutation group of the last \( m \) coordinates preserving the ordered set of weight vectors. The twisted hypergeometric cohomology group \( \mathcal{H}^m_{\text{ant}} \) is the antisymmetric part of the hypergeometric cohomology group with respect to the action of the same group.

The result of the construction is the complex linear space \((\mathcal{H}^m)^*\) and a \((\mathcal{H}^m)^*\)-valued differential equation (10) on \( \mathbb{C}^n \).

### 3.2 Representations of Lie Algebras and Knizhnik-Zamolodchikov Equation

As an example consider the Lie algebra \( g = sl_2(\mathbb{C}) \) of complex \( 2 \times 2 \)-matrices with the zero trace. \( g \) is generated by the standard generators \( e, f, h \) subject to the relations \([e, f] = h, [h, f] = -2f, [h, e] = 2e \).

Fix an invariant scalar product on \( g \) (the Killing form). Let \( \Omega \in g \otimes g \) be the tensor corresponding to the invariant scalar product (the Casimir operator).

Let \( L_1, \ldots, L_n \) be representations of \( g \), \( L = L_1 \otimes \cdots \otimes L_n \). Let \( \Omega_{jk} \) be the linear operator on \( L_1 \otimes \cdots \otimes L_n \), acting as the Casimir operator on \( L_j \otimes L_k \) and as the identity operator on the other factors. The **Knizhnik-Zamolodchikov (KZ) equation** on the \( L \)-valued function \( \phi(z_1, \ldots, z_n) \) is the system of the differential equations

\[
d\phi = \kappa^{-1} \sum_{1 \leq j < k \leq n} \Omega_{jk} \phi d(z_j - z_k)/(z_j - z_k),
\]

where \( \kappa \) is a complex parameter.

The KZ equation defines the integrable connection on the trivial bundle \( L \times \mathbb{C}^n \) with singularities over the diagonals. This connection has a remarkable property:
parallel translations of this connection commute with the action of $g$ on fibers. Thus the eigenspaces of the operators $e, f, h$ are invariant under parallel translations.

It turns out that the KZ equation restricted on suitable invariant subspaces coincides with suitable hypergeometric equations constructed above.

More precisely, let $\mathfrak{h} \subset g$ be the Cartan subalgebra generated by $h$. The Verma module of $g$ with the highest weight $\lambda \in \mathfrak{h}^*$ is the infinite dimensional representation of $g$ generated by one (vacuum) vector $v$ with the properties $ev = 0, hv = \langle h, \lambda \rangle v$. Verma modules are the simplest representations from which all finite-dimensional representations may be constructed. Let $M_1, \ldots, M_n$ be Verma modules with the highest weights $\lambda_1, \ldots, \lambda_n \in \mathfrak{h}^*$. Put $M = M_1 \otimes \cdots \otimes M_n, \lambda = \lambda_1 + \cdots + \lambda_n$. $M$ is the direct sum of the eigenspaces of the operator $h$, $M = \bigoplus_{m \geq 0} M_{\lambda - ma}$, where $M_{\lambda - ma}$ is the eigenspace with the eigenvalue $\langle h, \lambda - ma \rangle$, $\alpha \in \mathfrak{h}^*$ is the single positive root of $\mathfrak{sl}_2(\mathbb{C})$. The vacuum subspace $\text{Vac}_{\lambda - ma} \subset M_{\lambda - ma}$ is the subspace of all vectors annihilated by the operator $e$.

**Theorem** [SV, DJMM]. The vacuum subspace $\text{Vac}_{\lambda - ma}$ and the KZ equation restricted on it are canonically isomorphic, respectively, to the space $(\mathcal{H}_m^\omega)^*_{\text{ant}}$ dual to the twisted hypergeometric cohomology group, and to the differential equation with values in $(\mathcal{H}_m^\omega)^*_{\text{ant}}$ constructed by the hypergeometric construction from the projection $\mathbb{C}^{m+n} \to \mathbb{C}^n$, the linear space $\mathfrak{h}^*$, the Killing form on $\mathfrak{h}^*$ and the ordered set of the vectors $\lambda_1, \ldots, \lambda_n, -\alpha, \ldots, -\alpha \in \mathfrak{h}^*$.

**Corollary.** For a general $\kappa$ all solutions of the KZ equation with values in a tensor product of Verma modules are given by the hypergeometric functions.

An analogous picture has a place if the algebra $\mathfrak{sl}_2(\mathbb{C})$ is replaced by any Kac-Moody algebra, see [SV, DJMM, Ma, Ch, CF, L].

This theorem shows that the KZ equation has topological nature, it is just the Gauss-Manin connection of the simple fiber bundle.

### 3.3 Hypergeometric Functions in the Conformal Field Theory

The KZ equation was invented in the CFT. Its solutions describe $(n+1)$-point correlation functions on the Riemann sphere in the Wess-Zumino-Witten model [BPZ, KZ]. In the minimal models of the CFT correlation functions on the sphere have also integral representations in terms of the same configurations [DF]. For the first time integral representations of correlation functions in the CFT appeared in the works by Dotsenko and Fateev.

Any model of the CFT have a certain set of primary fields $\{\phi_n(z, \bar{z})\}$ and an operator algebra

$$\phi_n(z, \bar{z})\phi_m(0, 0) = \sum_p C^p_{nm} z^{-\Delta_n - \Delta_m - \Delta_p - \Delta_{np} - \Delta_{mp} - \Delta_{mn} - \Delta_{nm}} \phi_p(0, 0) + \cdots,$$

where the numbers $\Delta_n, \bar{\Delta}_n$ are the conformal dimensions of the field $\phi_n$, the numbers $C^p_{nm}$ are the structure constants of the model, in this expression the terms containing non-primary fields are omitted, see [BPZ]. The integral representations of the
correlation functions allowed one to calculate the structure constants of the minimal models of the CFT and of the Wess-Zumino-Witten model corresponding to \( \mathfrak{sl}_2(\mathbb{C}) \) [DF, Do]. It turns out that in these cases the structure constants are equal to certain alternating products of values of the gamma-function similar to that appearing in the determinant formula of Sect. 2.3. In these cases any structure constant was represented by a certain hypergeometric integral. Probably in other models of the CFT the structure constants are connected with suitable determinants of hypergeometric integrals which in their turn may be expressed as alternating products of values of the gamma-function.

3.4 Hypergeometric Functions and Quantum Groups

The hypergeometric integrals as functions of the parameters satisfy the KZ equation. The monodromy of the KZ equation is described in terms of quantum groups.

The monodromy representation of the KZ equation is the representation of the pure braid group on \( n \) strings in the group \( \text{Aut}(L_1 \otimes \cdots \otimes L_n) \) generated by analytic continuation of solutions along loops in the base. Denote it by \( \tau_k \). Algebraically we define another representation of the pure braid group.

According to Faddeev, Kulish, Reshetikhin, Sklylnin, Jimbo, Drinfeld [Dr1, J] the universal enveloping algebra \( U_\mathfrak{g} \) of the algebra \( \mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \) is deformed to the quantum universal enveloping algebra \( U_q \mathfrak{g} \) (the quantum group) depending on the complex parameter \( q \).

For the quantum group a comultiplication \( \Delta : U_q \mathfrak{g} \to U_q \mathfrak{g} \otimes U_q \mathfrak{g} \) is defined. If \( V_1, V_2 \) are representations of the quantum group then the comultiplication induces a representation structure on their tensor product. The representations \( V_1 \otimes V_2 \) and \( V_2 \otimes V_1 \) are isomorphic. The isomorphism is defined by the formula

\[
V_1 \otimes V_2 \overset{R}{\rightarrow} V_1 \otimes V_2 \overset{P}{\rightarrow} V_2 \otimes V_1,
\]

where \( P \) is the transposition of the factors and \( R \in U_q \mathfrak{g} \otimes U_q \mathfrak{g} \) is the distinguished element called the universal \( R \)-matrix of the quantum group.

Let \( V_1, \ldots, V_n \) be representations of the quantum group. Then the pure braid group acts on their tensor product. Namely let \( \sigma_1, \ldots, \sigma_{n-1} \) be the elementary braids shown in Fig. 5. To any braid \( \sigma_i \) assign the linear operator

![Fig. 5](image-url)
acting as \( PR \) on the \( i \)-th and \((i + 1)\)-th factors and as the identity operator on the other factors. These operators define the representation \( q \) of the pure braid group on \( n \) strings in \( \text{Aut}(V_1 \otimes \cdots \otimes V_n) \).

For general values of \( q \) the representation theory of the quantum group \( U_q g \) is the same as the representation theory of the algebra \( g \), any representation of the algebra \( g \) is deformed canonically to the representation of the quantum group. Let \( L_1, \ldots, L_n \) be representations of \( g \) and \( L_{1q}, \ldots, L_{nq} \) be their quantum deformation.

\textbf{Theorem} \([K, \text{Dr}2]\). The monodromy representation \( \tau_\kappa \) of the KZ equation with values in \( \text{Aut}(L_1 \otimes \cdots \otimes L_n) \) is equivalent to the \( R \)-matrix representation \( q, q = \exp(2\pi i/\kappa) \), with values in \( \text{Aut}(L_{1q} \otimes \cdots \otimes L_{nq}) \), if \( \kappa \) is not a rational number.

An analogous statement holds if \( \text{sl}_2(\mathbb{C}) \) is replaced by any Kac-Moody algebra.

This theorem is a wonderful statement. Given the differential equation of the CFT described in terms of the representation theory of the Lie algebra one considers its global characteristic, the monodromy representation defined by analytic continuation of solutions and one gets the representation defined by \( R \)-matrix of the quantum group which in its turn is defined as the deformation of the universal enveloping algebra of the initial Lie algebra.

It is interesting that new invariants of knots defined by Jones and others are constructed in terms of the same representations of the braid groups, see \([C2, \text{Jo}, L, \text{RT}]\).

The realization of the KZ equation as the Gauss-Manin connection allows one to make explicit the equivalence of the representations \( \tau_\kappa \) and \( q, q = \exp(2\pi i/\kappa) \). It turns out that for real \( z_1 < \cdots < z_n \) there is a canonical isomorphism of the quantum deformation of the vacuum subspace to the twisted homology group:

\[
\text{Vac}_{\lambda_{-na},q} \rightarrow H_m(\mathbb{C}^m \setminus \varepsilon(z), \mathcal{S}(z))_{\text{ant}},
\]

which sends the \( R \)-matrix representation to the monodromy representation of the Gauss-Manin connection. The equivalence of the Kohno-Drinfeld theorem is the composition

\[
\text{Vac}_{\lambda_{-na},q} \rightarrow H_m(\mathbb{C}^m \setminus \varepsilon(z), \mathcal{S}(z))_{\text{ant}} \xrightarrow{i}(\mathcal{H}^m_{\text{ant}})^* \rightarrow \text{Vac}_{\lambda_{-na}}.
\]

Thus a correspondence between a representation of Lie algebra and its quantum deformation is the correspondence between the twisted homology group and the dual space to the twisted hypergeometric cohomology group given by the integration of the hypergeometric forms.

The details of the proof of this statement and the analogous statement for arbitrary Kac-Moody algebra are being checked in \([SV4]\). On this subject see also \([L]\).
4. Concluding Remarks

Many interesting works on hypergeometric functions are not mentioned in this talk. Among them there are works by Deligne and Mostow, Beukers and Heckman on monodromy of hypergeometric functions, works by Gelfand, Kapranov, Zelevinsky on combinatorial description of differential equations on hypergeometric functions, works on $q$-analogs of hypergeometric functions, on difference equations on hypergeometric functions, works by Heckman and Opdam on hypergeometric functions and root systems.

The main problem is to unify numerous and remote investigations on hypergeometric functions into a united theory in which it might be possible to pass from Conformal Field Theory to Algebraic $K$-Theory by analytic continuation on parameters.

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