The Turing degrees $\mathcal{D}$ were introduced by Kleene and Post ([9], 1954) to isolate and study those properties of the subsets of the natural numbers $\mathbb{N}$ which are expressed purely in terms of relative computability. Intuitively, we form $\mathcal{D}$ by identifying any pair of subsets of $\mathbb{N}$ which are mutually computable and ordering the resulting equivalence classes by relative computability. The natural hierarchies of definability within arithmetic, analysis and higher fragments of set theory all have sharply focused images in $\mathcal{D}$.

We will focus our attention on the second order properties of the Turing degrees. We will phrase our discussion of the known results and especially of the techniques of their proofs within as large a context as possible, so to apply to degree structures based on other forms of relative definability, as well as to $\mathcal{D}$.

Formally, suppose that $A$ and $B$ are subsets of $\mathbb{N}$, henceforth called reals. We say that $A$ is Turing reducible to $B$ ($A \leq_T B$) if there is a computational procedure which takes an input $n$ from $\mathbb{N}$; over the course of its execution on input $n$, asks whether various numbers are in $B$; and, if it receives the correct responses to those questions, after finitely many computational steps returns the answer as to whether $n$ is an element of $A$. In other words, if we were given $B$ then we would be able to compute $A$. We say that $A$ and $B$ are Turing equivalent ($A \equiv_T B$) if $A \leq_T B$ and $B \leq_T A$. The Turing degrees $D$ are the $\equiv_T$-equivalence classes. The degree structure $\mathcal{D}$ associated with Turing reducibility is the partial ordering $\langle D, \leq_T \rangle$, the Turing degrees with the ordering inherited from $\leq_T$.

$\mathcal{D}$ is the most intensively studied degree structure but not the only one. Although we cannot introduce them all in detail, we will point out a few other examples. Among the many degree structures on the real numbers obtained by varying the notion of relative definability, we might mention the arithmetic degrees, the hyperarithmetic degrees, the $\Sigma_k$-degrees ($A \leq_{\Sigma_k} B$ if $A$ is in the least set which is $\Sigma_k$-admissible relative to $B$), the many-one degrees, the enumeration degrees, the $A_1^\infty$-degrees and the degrees of constructibility. Alternately, we can retain relative computability and vary the class of reals on which it acts. We can form the Turing degrees of the recursively enumerable sets, the Turing degrees...
of the $\Delta^0_3$-sets (i.e. the structure $\mathcal{D}(\leq_T \delta')$ on the degrees below $\delta'$) or the Turing degrees of the arithmetically definable sets. We will refer to these examples as local substructures of $\mathcal{D}$. For any degree $x$, we can also look at $\mathcal{D}(\geq_T x)$, the partial order of the degrees greater than or equal to $x$.

Extending our scope to include degree notions on other than sets of integers, we can also mention the Kleene degrees of sets of reals, the degrees of sets of reals modulo $3^E$, and the $\alpha$-degrees of subsets of an admissible ordinal $\alpha$. Similarly, we can form their local substructures such as $\alpha$-recursively enumerable $\alpha$-degrees or the Kleene degrees of the $\Pi^1_1$-subsets of the continuum.

To some extent, in our discussion of these structures, we will follow the historical order in which their properties where discovered. We will focus our discussion on the Turing degrees since historically $\mathcal{D}$ has been the proving ground for the analysis of degree structures. To begin with the obvious: there is a least degree consisting of the computable reals; since there are only countably many computational procedures, each Turing degree is countable and has only countably many predecessors, any two degrees $x$ and $y$ have a least upper bound $x \vee y$ (so $\mathcal{D}$ is an upper-semi-lattice with a natural operation of join) and the cardinality of the whole structure is the same as the continuum. To continue beyond these immediate observations we must directly analyze the notion of relative computability.

In the first section, we will discuss the early results about the Turing degrees. For the most part, these were to the effect that $\mathcal{D}$ has a rich existential theory, showing that $\mathcal{D}$ plays various roles as a universal object. In the second section, we will describe the subsequent results which simultaneously limit $\mathcal{D}$'s universal role and reveal its detailed structure. In the final sections, we will discuss some recent results and conjectures of Slaman and Woodin. If true, the conjectured structural theorem would provide a complete logical characterization of $\mathcal{D}$. We prove the conjecture for the hyperarithmetic degrees and for $\mathcal{D}$ with finitely many additional parameters.

§1. The Embedding Theorems of the Late 1950s, 1960s and Early 1970s

Partial Order Embeddings

In their 1954 paper [9], Kleene and Post used a Baire category argument, one involving a primitive form of Cohen forcing, to show that there are sets of incomparable Turing degree. In fact, Kleene and Post proved that every finite partial order can be embedded in $\mathcal{D}$. Sacks ([18], 1963) extended the Kleene-Post embedding theorem to show that every countable partial ordering, and even every one of size $\aleph_1$ which is locally countable, could be embedded in $\mathcal{D}$.

Sacks conjectured:

(1) (Sacks ibid., 1963) **Conjecture:** A partially ordered set $P$ is imbeddable in the (Turing) degrees if and only if $P$ has cardinality at most that of the continuum and each member of $P$ has at most countably many predecessors.
The Kleene-Post and Sacks partial order embedding results indicated a universal quality of \( \mathcal{D} \). Sacks expressed a belief in this quality by conjecturing that the strongest possible purely existential property would be satisfied by \( \mathcal{D} \). (Surprisingly, this conjecture is still open.) Furthermore, the algebraic properties of \( \mathcal{D} \) seemed to be based more in the nature of relative definability than in specific properties of computability. The proofs of the embedding theorems apply to any of the above mentioned degree notions on the reals which are countably based, i.e. excluding only the \( A^1_2 \)-degrees and the degrees of constructibility.

**Initial Segments**

Say that a subset \( I \) of \( \mathcal{D} \) is an **ideal** if it is an initial segment and closed under join. \( I \) is a **principal ideal** if in addition it has a greatest element. Spector ([23], 1956) answered a question of Kleene-Post [9] by constructing a two element ideal, i.e. a minimal nontrivial degree. Spector's theorem rules out the potential classification of the principal ideals in \( \mathcal{D} \) (as the countable model of an \( \aleph_0 \)-categorical theory). For example, there are two nontrivial ideals which are not isomorphic. Spector's method could be extended to build many examples of isomorphism types of initial segments in \( \mathcal{D} \). Prompted by Spector's results, Sacks made the following conjecture.

(2) (Sacks [19], 1966) **Conjecture:** \( S \) is a finite, initial segment of the degrees if and only if \( S \) is order isomorphic to a finite, initial segment of some upper-semi-lattice with a least member.

Sacks's conjecture postulated that another universal property would hold of \( \mathcal{D} \): the initial segments of \( \mathcal{D} \) would realize all the finite possibilities. Extending work of Thomason and Lachlan, Lerman ([12], 1971) confirmed this conjecture. Lachlan and Lebeuf ([11], 1976) showed that every initial segment of a countable upper-semi-lattice with least element is isomorphic to an initial segment of \( \mathcal{D} \). Ultimately, Abraham and Shore ([1], 1986) showed that every initial segment of an upper-semi-lattice which is locally countable and of cardinality \( \aleph_1 \) is isomorphic to an initial segment of \( \mathcal{D} \). Groszek and Slaman ([4], 1983) showed that the Abraham-Shore theorem is best possible; it is independent of \( ZFC + 2^{\omega} > \aleph_1 \) whether every initial segment of an upper-semi-lattice which is locally countable and of cardinality \( \aleph_2 \) can be embedded in \( \mathcal{D} \) as an upper-semi-lattice.

The proofs of these theorems, while more technically demanding than the proofs of the embedding theorems, could still be applied to a wide range of countably based degree structures. The enumeration degrees are the only notable exception.

From the progress on the Sacks conjectures, degree theorists speculated whether \( \mathcal{D} \) might have an algebraic characterization, or at least occupy a distinguished position among the upper-semi-lattices.
Questions from the 1960s

Workers in the field faced several fundamental questions. We pose them here for \( \mathcal{D} \), but they are completely general. Is there a global structure theory for \( \mathcal{D} \)? Is the theory of \( \mathcal{D} \) specific to Turing reducibility or is it applicable to a general class of degree structures? Is the structure of \( \mathcal{D} \) tied to the continuum or is it reflected in the local substructures of \( \mathcal{D} \)?

We recall some specific questions from that time.

(3) (Sacks [19], 1966) Is the theory of \( \mathcal{D} \) decidable?
(4) (Sacks ibid., 1966) Are the Turing degrees and the Turing degrees of the arithmetic sets (\( \mathcal{A} \)) elementarily equivalent?
(5) (Rogers [17], 1967) For a degree \( a \), let \( \mathcal{D}(\geq_T a) \) denote the restriction of \( \mathcal{D} \) to those degrees above \( a \). Is it the case that for all \( a \) and \( b \), \( \mathcal{D}(\geq_T a) \sim \mathcal{D}(\geq_T b) \)?
(6) (Rogers ibid., 1967) Is there a nontrivial automorphism of \( \mathcal{D} \)? (If not then we say that \( \mathcal{D} \) is rigid.)

Rogers defined a relation \( R \) on degrees to be absolutely definable in \( \mathcal{D} \) if \( R \) is invariant under all automorphisms of \( \mathcal{D} \).

(7) (Rogers ibid., 1967) Are the Turing jump and the relation recursively enumerable in absolutely definable in \( \mathcal{D} \)? In general, which relations are absolutely definable in \( \mathcal{D} \)?

§2. Coding and Definability Theorems of the 1970s and Early 1980s

The results of the 1970s to the mid 1980s ruled out any reasonable understanding of the Turing degrees in algebraic terms. But then, the exact properties of \( \mathcal{D} \) that make it algebraically intractable were used during this period to settle almost all of the Sacks and Rogers questions. Their solutions illustrate the complexity of \( \mathcal{D} \): \( \mathcal{D} \) is not decidable; the theory of \( \mathcal{D} \) is not equal to the theory of the Turing degrees of the arithmetic sets; there are \( a \) and \( b \) such that \( \mathcal{D}(\geq_T a) \) and \( \mathcal{D}(\geq_T b) \) are not isomorphic. Rogers’s question whether the jump is definable was only recently settled (the jump is definable), but the techniques in its solution were steadily developed through this period. The only questions that remain open are whether \( \mathcal{D} \) is rigid and to give a classification of which relations are definable in \( \mathcal{D} \).

The trend is for the global properties of degree structures to follow those of the Turing degrees. The notable exception has been the many-one degrees. There, Ershov ([3], 1975) and Paliutin ([16], 1975) succeeded in obtaining an algebraic characterization for the partial ordering of the many-one degrees. The reader might also see Odifreddi ([15], 1989).
A Cone of Minimal Covers

One of the earlier questions to fall was whether $\mathcal{D}$ is elementarily equivalent to $\mathcal{A}$. Jockusch and Soare ([8], 1970) showed that there is no arithmetic degree $x$ such that every $y$ above $x$ is a minimal cover of some $z$ less than $y$. Jockusch ([5], 1973) showed that there is a nonarithmetic degree with the above property. Thus, the Turing degrees of the arithmetic sets are not elementarily equivalent to $\mathcal{D}$.

Coding in $\mathcal{D}$ and Undecidability

A primary ingredient in the work during this period was the proving and exploiting of coding theorems.

Definition. Suppose that $\mathcal{U}$ is a model of the finite language $\mathcal{L}$ and $\bar{p}$ is a finite sequence of parameters from $\mathcal{D}$. $\mathcal{U}$ is coded by $\bar{p}$ in $\mathcal{D}$ if there is an isomorphic image of $\mathcal{U}$ whose universe, relations, functions, constants and quantifiers are all first order definable in the language of $\mathcal{D}$ with additional symbols for the parameters $\bar{p}$.

A disparate sequence of coding schemes preceded the one which we have isolated below, see Simpson ([20], 1977) or Nerode-Shore ([13], 1979). Say that a relation $R$ is countable if there is a countable subset of the degrees such that all of the solutions to $R$ come from that set.

Coding Lemma (Slaman-Woodin [21], 1986). For any countable relation $R$ on degrees there are parameters $\bar{p}$ such that $R$ is definable in $\mathcal{D}$ from $\bar{p}$.

The coding lemma is uniform in the following sense. For each $n$, there is a fixed first order formula $\varphi$ such that for every countable $n$-ary relation $R$ on $\mathcal{D}$ there is a sequence $\bar{p}$ so that $R$ is defined from $\bar{p}$ using the formula $\varphi$ in $\mathcal{D}$. The proof of the coding lemma uses finite conditions to construct $\bar{p}$ from $R$; consequently, it does not use any machinery that is special to the Turing degrees. It applies or can be modified to apply to a very broad class of degree structures, even to some without minimal degrees such as the enumeration degrees, see Slaman-Woodin ([22], to appear).

We can use the coding lemma to present the solution to Sacks's question whether $\mathcal{D}$ is decidable. By the coding lemma, we can both code the standard model of arithmetic and also define the collection of codes of standard models. In addition, we can interpret second order quantifiers over a coded countable model by quantifiers over sequences in $\mathcal{D}$ which define unary relations. Thus, (Simpson [20], 1977) there is an interpretation of second order arithmetic in the first order theory of $\mathcal{D}$. Lachlan ([10], 1968) gave the original solution to Sacks's question but only showed that the theory of $\mathcal{D}$ is not recursive. Simpson calculated the degree of the first order theory of $\mathcal{D}$.)
Shore's Program

The next developments which we will discuss were initiated by Nerode and Shore ([14], 1980) and pursued extensively by Shore and his collaborators.

Suppose that \( R \) is a relation on degrees. We note that the degrees which are produced by the coding lemma to define \( R \) in \( D \) are recursion theoretically close to \( R \). By this we mean the following. Let \( X \) be a real of degree \( x \). If \( R \) is recursively presented relative to \( X \) then there is a sequence of degrees \( \bar{p} \) which codes \( R \) in \( D \) and is arithmetic in \( x \). As a special case, there is a sequence \( \bar{p}_X \) of sets which is arithmetically definable from \( X \) and whose degrees code (in \( D \)) an isomorphic copy of the standard model of arithmetic with a unary predicate for \( X \). Thus, the set \( X \) is coded in \( D \) by parameters which are near its degree. Furthermore, if \( X \) is sufficiently complicated, say above 0', then a sequence of parameters whose degrees code \( X \) in \( D \) can be found recursively in \( X \), that is below \( x \) in \( D \).

The next step was to find a notion of neighborhood which would be first order definable in \( D \) and link an arbitrary degree \( x \) to the reals coded in its neighborhood. The Jockusch-Soare theorems pointed in the correct direction. Jockusch and Soare defined a filter in \( D \) and proved that it is disjoint from the degrees of the arithmetic sets. Their theorem suggested that there might be a related definable filter whose complement was exactly the arithmetic degrees or even the degrees below 0'.

The search for such a filter lead to the development of the theory of \( REA \)-operators, see Jockusch-Shore ([7], 1984). Recently, Cooper ([2], 1990) combined the analysis of \( REA \)-operators with his study of the degrees of differences of recursively enumerable sets to prove that the Turing jump is definable in the Turing degrees. In fact, Cooper proved the stronger theorem that the relation \( x \) is recursively enumerable in and above \( y \) is definable in \( D \) (solving Roger's question). We remark that this approach is specific to the Turing degrees. The proofs directly exploit the way that Turing reducibility can be recursively approximated.

We list some applications, in the strongest form obtained by these techniques. (Unfortunately, we do not have the space to describe the historical development and give credit where it is due.)

**Theorem.**

- (Jockusch-Shore [7], 1984) For all \( x \) and \( y \), if \( D(\geq x) \rightarrow D(\geq y) \) then \( x \) and \( y \) have the same arithmetic degree.

- (Cooper [2], 1990) If \( \pi \) is an automorphism of \( D \) then \( \pi \) is the identity on all degrees greater than or equal to \( 0'' \).

- (Cooper ibid.) If \( R \) is a relation on the reals which are Turing above \( 0'' \), \( R \) is invariant under Turing degree and \( R \) is definable in second order arithmetic then the relation induced by \( R \) is definable in \( D \).

The last two examples were known to follow from the definability of the jump, see Nerode-Shore ([14], 1980).
A Shortcoming of Method

The three structural properties of \( \mathcal{D} \) we isolated above were proven above by following some practical advice: examine the neighborhood of \( x \) and recover some information about which reals belong to \( x \). In fact, we will exactly recover the set of representatives of \( x \) provided that the set of representatives of \( x \) is coded in the neighborhood of \( x \) and that the neighborhood of \( x \) and the method employed decode the set of its representatives are uniformly recursive in \( x \). Since the statement of the Turing order between reals involves \( \Sigma^0_3 \) and \( \Pi^0_3 \) predicates, \( x \) has to provide \( 0'' \), the information necessary to evaluate these predicates. Consequently, our conclusions for the Turing degrees have all been confined to the realms of sufficiently large degrees or arithmetic equivalence between Turing degrees. Even using Cooper's theorem to get the strongest possible bound on the orbit of \( 0' \), these methods have (so far) been confined by this three quantifier limit.

We should remark that this shortcoming does not apply to the degrees of constructibility. In that notion of degree, the partial ordering of the degrees of constructibility is below \( x \) is just another set constructed from \( x \). In fact, with reasonable set theoretic hypotheses the techniques we have touched upon are sufficient to give a complete analysis of the global theory of the constructibility degrees, in the sense of the next section.

§3. Recent Results

In this section, we will discuss some recent work of Slaman and Woodin ([22], to appear). Unless we specifically indicate otherwise, all of the following results, conjectures and even remarks are drawn from that source.

Assignment of Representatives

Suppose that \( \mathcal{E} \) is a degree structure on a set of reals, possibly the set of all reals. Say that \( \mathcal{E} \) is determined by the equivalence relation \( \equiv_E \) and the ordering \( \leq_E \).

**Definition.** \( \mathcal{E} \) has a first order assignment of representatives if
(1) for every \( \mathcal{E} \)-degree \( x \) there is a sequence \( p \) from \( \mathcal{E} \) which uniformly codes a representative of \( x \) (say using the formulas \( \varphi_1, \ldots, \varphi_k \));
(2) the relation \( p \) codes a representative of \( x \) is an \( \mathcal{E} \)-definable relation (say by the formula \( \psi \)).

We say that \( \mathcal{E} \) has a first order assignment of representatives in parameters if the same conditions hold as above except that we allow \( \varphi_1, \ldots, \varphi_k \) and \( \psi \) to mention parameters from \( \mathcal{E} \).

Suppose that \( \mathcal{E} \) has a first order assignment of representatives. If \( \mathcal{E} \) is definable in second order arithmetic and is a degree structure on all of the reals then we say that \( \mathcal{E} \) is biinterpretable with second order arithmetic. If \( \mathcal{E} \) is definable in first order arithmetic and is a degree structure on a uniformly arithmetic set of reals
then we say that $\mathcal{E}$ is biinterpretable with first order arithmetic. Similarly, we can define biinterpretability with parameters.

If $\mathcal{E}$ is biinterpretable with the standard model second order arithmetic then all of the logical properties of $\mathcal{E}$ can be reduced to the reals. We will list some examples. Others will undoubtedly occur to the reader.

Assuming $\mathcal{E}$ is biinterpretable with second order arithmetic:

- A relation is definable in $\mathcal{E}$ if and only if it is induced by an $\equiv_k$-invariant relation that is definable in second order arithmetic.
- $\mathcal{E}$ has no nontrivial automorphism, i.e. $\mathcal{E}$ is rigid.

Assuming that $\mathcal{E}$ is biinterpretable with second order arithmetic, we sketch an example proof. The standard model of first order arithmetic is rigid because each number is definable. Consequently, the standard model of second order arithmetic is also rigid since each subset of $\mathbb{N}$ is determined by its elements. The fact that no representative of a degree can be moved by an automorphism of the standard model of second order arithmetic implies that an automorphism of $\mathcal{E}$ can only move a degree to another one with the same set of representatives. Of course, this last condition is another way of saying that the only automorphism of $\mathcal{E}$ is the identity.

If we allow parameters we obtain the boldface versions of the above conclusions.

Assuming $\mathcal{E}$ is biinterpretable with second order arithmetic using the parameters $\hat{e}$:

- A relation is definable from finitely many parameters in $\mathcal{E}$ if and only if it is induced by an $\equiv_k$-invariant relation on reals that definable from finitely many parameters in second order arithmetic, i.e. is induced by a projective relation.
- Any automorphism of $\mathcal{E}$ is determined by its action on $\hat{e}$. That is, $\hat{e}$ is an automorphism base.

Just knowing that $\mathcal{E}$ has a finite automorphism base gives us some information about its group of automorphisms. For example, if $\mathcal{E}$ is a degree structure on the reals then the cardinality of the orbit of any finite sequence from $\mathcal{E}$ is bounded by the cardinality of the continuum. Thus, if $\mathcal{E}$ has a finite automorphism base then it has at most continuum many automorphisms. Similarly, if $\mathcal{E}$ is a degree structure on a countable set of reals and has a finite automorphism base then the automorphism group for $\mathcal{E}$ is countable.

A Finite Automorphism Base

We will now sketch a heuristic approach to proving that a degree structure on the reals has a finite automorphism base. The method directly applies to the specific structures of the Turing degrees, the arithmetic degrees, the hyperarithmetic degrees, the $\Sigma_k$-admissible degrees, the PTIME Turing degrees and the enumeration degrees.
Let $\mathcal{D}$ denote any of the above degree structures on the reals. In the next few paragraphs we will be measuring definability in terms of a number of jumps. The interpretation of jump depends on the interpretation of $\mathcal{D}$. By arithmetic in, we mean below some finite number of jumps. The reader may comfortably imagine that we are discussing the Turing degrees with the Turing jump.

For the first step, we prove the coding lemma with its full uniformity. In particular, there is a degree $z_0$ such that if $x$ is above $z_0$ then all the representatives of $x$ are coded below $x$. Some examples for $z_0$ are $0''$ in the Turing degrees, $0^{\omega}$ in the arithmetic degrees and the degree of Kleene’s $\mathcal{C}$ in the hyperarithmetic degrees. Let $z_0$ be fixed.

We continue with some countable algebra for $\mathcal{D}$.

**Definition.** Let $I$ be a countable ideal in $\mathcal{D}$ and let $\varphi$ be an automorphism of $I$. We say that $\varphi$ is persistent if for every $x$ in $\mathcal{D}$ there are an ideal $J$ and an automorphism $\varphi^*$ of $J$ such that $x \in J$, $I \subseteq J$ and $\varphi^*$ agrees with $\varphi$ on $I$.

Next we prove that every persistent countable automorphism extends to an automorphism of $\mathcal{D}$. The proof employs a generalization of an insight due to Odifreddi and Shore: the coding lemma can be used to show that the restriction of an automorphism of $\mathcal{D}$ to a countable ideal is recursion theoretically close to any uniform upper bound on that ideal. Specifically, we show that if $\varphi$ is a persistent automorphism of the ideal $I$ then $\varphi$ is arithmetic in any upper bound of $I$.

We now introduce some metamathematical methods. We know that there is a nontrivial automorphism of $\mathcal{D}$ if and only if there is a nontrivial countable automorphism that is persistent. The latter condition is upwards absolute between well-founded models of ZFC.

Applying results of Slaman-Woodin ([21], 1986), we obtain the following theorem.

**Theorem.** $\mathcal{D}$ is rigid if and only if $\mathcal{D}$ is biinterpretable with second order arithmetic.

Continuing the metamathematical discussion, let $V$ denote the universe of sets. Let $V[\mathcal{D}]$ be a generic extension of $V$. Using the absoluteness theorem, we show that if $\pi$ is an automorphism of $\mathcal{D}^V$, the degrees in $V$, then $\pi$ lifts to an automorphism $\pi^*$ of $\mathcal{D}^V[\mathcal{D}]$, the degrees in the generic extension. By moving to a generic extension of $V$, we can use the definition of forcing to analyze $\pi^*$. In particular, if $\mathcal{D}$ is generic with respect to the partial order to add $\omega_1$ Cohen reals to $V$ then $\pi^*$ is represented as a continuous function on the set of generic reals. In fact, we can use our proof that $\pi(x)$ is close to $x$ to show the following. There is a recursive functional $\{e\}$ and an integer $n$ such that if $G$ is a Cohen generic real over $V$ then the degree of $\{e\} (G \oplus (n^{-1}(Z_0))^n)$ is $\pi^*$ of the degree of $G$. Here we let $Z_0$ denote a representative of $z_0$, $n^{-1}(Z_0)$ denote a representative of $n^{-1}(z_0)$ and $(n^{-1}(Z_0))^n$ denote the $n$th jump of $n^{-1}(z_0)$.

Our next step is to extract a representation of $\pi$ on a comeager set of reals in $V$ from the representation of $\pi^*$ on the comeager set of generic reals in $V[\mathcal{D}]$. 
We prove that the same representation of \( \pi \) (using the functional \( \{ e \} \) relative to \( \pi^{-1}(Z_0)^{(6)} \)) holds on the set \( C \) of reals which are sufficiently generic relative to \( \pi^{-1}(z_0) \). The level of genericity required is only finitely many jumps, in the sense of the reducibility determining \( \mathcal{D} \). Furthermore, if there is one \( G \) in \( C \) such that \( G \) and \( \{ e \} (G \Theta (\pi^{-1}(Z_0))^{(6)}) \) have the same degree then \( \pi \) is the identity on the degrees represented in \( C \).

In ([6], 1981), Jockusch and Posner show that the Turing degrees represented by any comeager set of reals generate the Turing degrees under the operations of meet and join. Their argument is completely general, so we may conclude that \( C \) is an automorphism base.

Now we can make two observations.

First, suppose that \( \pi(z_0) = z_0 \). Then the set \( C \) consists of those reals which are arithmetically generic relative to \( z_0 \). Let \( g_0 \) be any degree of an element of \( C \). If \( \pi \) also maps \( g_0 \) to \( g_0 \) then \( \pi \) must be the identity on \( C \) and therefore \( \pi \) must be the identity on all of \( \mathcal{D} \). Thus, \( \{ z_0, g_0 \} \) is a finite automorphism base for \( \mathcal{D} \).

Second, from the arithmetic representation of \( \pi \) on \( C \) relative to \( \pi^{-1}(Z_0) \) we can find a function arithmetic in \( \pi^{-1}(Z_0) \) which represents \( \pi \) on all the reals. Consequently, the fact that the rigidity of \( \mathcal{D} \) is absolute may be explained by the fact that every automorphism of \( \mathcal{D} \) is arithmetically definable in a real parameter.

**Biinterpretability with Parameters**

The fact that \( \mathcal{D} \) has a finite automorphism base can be combined with the analysis of persistent automorphisms to show that \( \mathcal{D} \) is biinterpretable with second order arithmetic in parameters.

Let \( Z_0 \) and \( G_0 \) denote representatives of the degrees \( z_0 \) and \( g_0 \) above. Suppose that \( \psi \) is a map from the reals onto \( \mathcal{D} \) which induces an automorphism on \( \mathcal{D} \), i.e. \( \psi \) is degree invariant, preserves order in the sense of \( \mathcal{D} \)-reducibility and has distinct values on reals of distinct degree. If \( \psi \) maps \( Z_0 \) to \( z_0 \) and \( G_0 \) to \( g_0 \) then \( \psi \) must induce the identity automorphism. In other words, \( \psi \) must be an assignment of representatives to \( \mathcal{D} \). Let the parameters \( \hat{p}_{Z_0} \) code \( Z_0 \) and \( \hat{p}_{G_0} \) code \( G_0 \). With some finesse, we can use the coding lemma to express the following condition in \( D \): *There is a persistent countable assignment of representatives sending \( Z_0 \) to \( z_0 \), \( G_0 \) to \( g_0 \) and \( X \) to \( x \).* By the characterization of persistent countable automorphisms as restrictions of global automorphisms, this statement is equivalent to one saying that there is a map from the reals to \( \mathcal{D} \) with the values as above that induces an automorphism. Of course, this automorphism is the identity. Thus, the statement above defines \( X \) is a representative of \( x \) as expressed in the codes for the real \( X \). Consequently, \( \mathcal{D} \) is biinterpretable with second order arithmetic using the parameters \( \hat{p}_{Z_0}, z_0, \hat{p}_{G_0} \) and \( g_0 \).

**Theorem.** The structures of the Turing degrees, the arithmetic degrees, the hyper-arithmetic degrees, the \( \Sigma_k \)-admissible degrees, the \( \text{PTIME} \) Turing degrees and the enumeration degrees are all biinterpretable with second order arithmetic in parameters.
Special Arguments for the Turing, Arithmetic, Hyperarithmetic and $\Sigma_k$-Admissible Degrees

In this section, we will discuss results obtained using special properties common to the Turing, Arithmetic, Hyperarithmetic and $\Sigma_k$-admissible reducibilities. We should note that these arguments do not intersect with those of Jockusch-Shore and Cooper and provide new proofs of the three structural properties of the Turing degrees mentioned above.

Since the proofs here are more technically involved than the ones that we have discussed so far, we have to resign ourselves to merely stating results. We can only say that the proofs of these results involve a direct analysis how a continuous function must behave to represent the restriction of a degree structure automorphism to a comeager set.

By the remark of the previous section, for all of our degree structures of interest, rigidity is equivalent to biinterpretability with second order arithmetic. Thus, the central problem for any of these structures is whether it has a nontrivial automorphism.

**Theorem.**

1. The hyperdegrees are rigid (and thus are biinterpretable with second order arithmetic.) Similarly, all of the $\Sigma_k$-admissible degrees structures are biinterpretable with second order arithmetic.
2. Any automorphism of the arithmetic degrees is the identity above $0^{\omega}$.  
3. Any automorphism of the Turing degrees is the identity above $0''$.

**Corollary.** There are only countably many automorphisms of the Turing degrees. Any automorphism of the Turing degrees is represented by an arithmetically definable function on reals.

A similar corollary holds for the arithmetic degrees; any automorphism is hyperarithmetically representable.

Special Arguments for the Turing Degrees

We now restrict $\mathcal{D}$ to denote the Turing degrees. We also drop any restraint against using very special properties of $\mathcal{D}$. In particular, we will make full use of Cooper’s theorem that the relation $x$ is recursively enumerable in and above $y$ is definable in $\mathcal{D}$.

Earlier, we sketched a metamathematical proof that any automorphism of $\mathcal{D}$ is arithmetically definable. We can also give more traditional, purely recursion theoretic proof of this fact. Of course, this proof is a more difficult local version of its metamathematical progenitor. With the sharper argument, we can replace the full structure of $\mathcal{D}$ by any ideal in $\mathcal{D}$ which has $0^{(7)}$ as element.
Theorem. Suppose the $I$ is an ideal in $D$ and $0^{(7)} \in I$.

1. If $\pi : I \to I$ then $\pi$ is represented by an arithmetically definable function on reals.

2. $I$ is biinterpretable in parameters with the fragment of second order arithmetic in which the second order quantifiers range over the reals whose degrees lie in $I$.

As a corollary to the theorem, we can demonstrate a connection between the existence of a local automorphism and a global one. By the Kleene basis theorem, if an arithmetic function does not represent an automorphism of $D$ then there is a counter-example which is recursive in Kleene's $0$. Thus, we can conclude that any automorphism of $D(\leq_T 0')$ extends to an automorphism of $D$.

A Concrete Automorphism Base for the Turing Degrees

We already know that there is a finite automorphism base for $D$. The question arises as to how concrete can we make the base elements. Outside of proving that the Turing degrees are rigid, we can give the best possible result. We show that there is a finite set of recursively enumerable degrees which is an automorphism base for $D$.

Our approach is to provide a generating family of first order formulas such that the smallest set including the recursively enumerable degrees and closed under definition by these formulas includes the degrees of $0''$ (i.e. $\omega_0$) and of a real $G$ which is sufficiently generic for the pair of degrees to form a base. We isolate an result which we prove along the way which is of independent interest.

Theorem.

1. $D(\leq_T 0')$ is biinterpretable with first order arithmetic in parameters. In fact, we may take the parameters to be recursively enumerable degrees.

2. Any automorphism of $D(\leq_T 0')$ is arithmetically definable.

Ultimately, we prove that there is a finite set $F$ of recursively enumerable degrees such that $D$ is biinterpretable with second order arithmetic in the parameters from $F$. In fact, the analogous theorem holds of any ideal $I$ such that $0^{(7)} \in I$. Thus, we have the following theorem.

Theorem.

1. The recursively enumerable degrees are an automorphism base for $D$. Further, $D$ is biinterpretable with second order arithmetic using recursively enumerable parameters.

2. If the recursively enumerable degrees are rigid then $D$ is rigid.

3. If $D(\leq_T 0')$ is rigid then $D$ is rigid.

It is open whether every automorphism of the recursively enumerable degrees extends to one of $D(\leq_T 0')$ or even to $D$. Any result along this line would be valuable.
§4. Conjectures

We make the following conjectures jointly with W. H. Woodin.

We say that a real number $G$ is 1-generic if for every recursively enumerable set $S$ of finite Cohen conditions either there is an element of $S$ which is satisfied by $G$ or there is a neighborhood condition satisfied by $G$ which is incompatible with all the elements of $S$.

**Conjecture I.** If $I$ is an ideal in the Turing degrees such that there is the degree of a 1-generic real in $I$ then $I$ has a first order assignment of parameters.

In particular, we believe that the partial ordering of the Turing degrees is biinterpretable with second order arithmetic and that $\mathcal{D}(\leq_T 0')$ is biinterpretable with first order arithmetic.

**Conjecture II.** The partial ordering of recursively enumerable degrees is biinterpretable with the standard model of first order arithmetic.

We end with a question. Is there a general proof of rigidity and equivalently of biinterpretability with second order arithmetic that is based on simple properties of Turing reducibility and applies to a wide range of degree structures?

**References**

5. Jockusch, C. A., Jr.: An application of $\Sigma^0_4$ determinacy to the degrees of unsolvability. J. Sym. Logic 38 (1973) 293–294