Identities of Associative Algebras

Alexander R. Kemer

Ulyanovsk Branch of Moscow State University, Str. Tolstogo 42, Ulyanovsk 432700, USSR

Preliminaries

Let $F$ be field, $F\langle X \rangle$ the free associative algebra over $F$ generated by a countable set of variables $X$. One may consider an element from $F\langle X \rangle$ as a polynomial in non-commutative variables from $X$.

**Definition 1.** We shall say that an associative algebra $A$ over $F$ satisfies an identity $f(x_1, \ldots, x_n) = 0$, where $f = f(x_1, \ldots, x_n) \in F\langle X \rangle$, if for arbitrary elements $a_i \in A$ the equality $f(a_1, \ldots, a_n) = 0$ is always valid in $A$.

**Definition 2.** The set of all polynomials $f \in F\langle X \rangle$ such that $A$ satisfies the identity $f = 0$ is said to be the ideal of identities of the algebra $A$. We shall denote this ideal by $T[A]$. An ideal of $F\langle X \rangle$ which is an ideal of identities of some algebra is called a T-ideal.

**Definition 3.** An Algebra $A$ is said to be PI-algebra if it satisfies a non-trivial identity (i.e. $T[A] \neq 0$).

The structure theory for PI-algebras is well developed. Some results of this theory are classic now. One of them is Kaplansky's theorem which asserts that a primitive PI-algebra is finite dimensional over its centre. Another example is the theorem of Nagata-Higman which asserts that any algebra over a field of zero characteristic satisfying identity $x^n = 0$ is nilpotent.

In 1957 A.I. Shirshov proved his famous Height Theorem:

**Theorem** (A.I. Shirshov [1]). For any finitely generated PI-algebra $A$ there exist a number $h$ and elements $a_1, \ldots, a_n \in A$ such that elements

$$a_{i_1}^{s_1} \cdots a_{i_k}^{s_k}, \quad k < h$$

generate $A$ as a space.

The Height Theorem implies the positive solution of Kurosh's problem for PI-algebras: A finitely generated algebraic PI-algebra is finite dimensional.
Although Kurosh's problem for PI-algebras was solved earlier in 1948 by I. Kaplansky, Shirshov's approach shows clearly the contributions of both conditions (PI and algebraicity) in order that algebra be finite dimensional. The Height Theorem shows also that finitely generated PI-algebras are close to finite dimensional algebras.

Shirshov's Height Theorem was the first result which gave some information about identities of PI-algebras.

This paper is also devoted to the study of the structure of identities of associative algebras.

1. Finitely Generated PI-Algebras

First of all we recall the famous result in PI-theory which was proved by A. Braun in 1982.

**Theorem** (A. Braun [2]). The radical of a finitely generated PI-algebra is nilpotent.

This theorem has the following important corollary:

**Corollary.** A finitely generated PI-algebra satisfies for some $n$ the identity

$$\sum_{\sigma \in S(n)} (-1)^{\sigma}x_{\sigma(1)}y_1x_{\sigma(2)}y_2 \cdots y_{n-1}x_{\sigma(n)} = 0$$

(1)

where $S(n)$ is the symmetric group of degree $n$; $(-1)^{\sigma} = 1$ if $\sigma$ is an even permutation, $(-1)^{\sigma} = -1$ if $\sigma$ is odd.

The identity (1) is called the Capelli identity of $n$-th order.

It is well known that if $\text{char } F = 0$ then the Capelli identities “distinguish” finitely generated PI-algebras from infinitely generated algebras.

**Theorem.** Let an associative PI-algebra $A$ over a field of zero characteristic satisfy the Capelli identity of some order. Then there exists a finitely generated PI-algebra $B$ such that $T[A] = T[B]$.

If $\text{char } F \neq 0$ then the conclusion of this theorem is not true. We shall discuss this case later.

The main result about identities of finitely generated PI-algebras is the following theorem which was proved by the author in 1988.

**Theorem 1** ([3, 4]). For any finitely generated PI-algebra $A$ over an infinite field $F$ there exists a finite dimensional $F$-algebra $C$ such that $T[A] = T[C]$.

In other words finitely generated PI-algebras cannot be distinguished from finite dimensional algebras in the language of identities.
If $F$ is a finite field then the conclusion of Theorem 1 is not valid, but Theorem 1 gives a full information about identities which are homogeneous with respect to every variables.

One may consider Theorem 1 as a theorem on classification of finitely generated PI-algebras in the language of identities.

**Definition 4.** Let $T$ be a $T$-ideal in the free algebra $F\langle X \rangle$. The factor algebra $F\langle X \rangle / T$ is said to be a relatively free algebra generated by the set of variables $X$.

**Definition 5.** An algebra over a field $F$ is called representable if it can be embedded into a matrix algebra of finite order over some extension of the basic field $F$.

Theorem 1 has the following important corollary:

**Corollary.** Any relatively free PI-algebra of finite rank is representable (the basic field is infinite).

Since the proof of Theorem 1 has been published we give the main ideas of this proof.

Let $A$ be a finitely generated PI-algebra. Using the theorem of Braun and Levin's theorem it is easy to prove that

$$T[A] \supseteq T[C]$$

for some finite dimensional algebra $C$. That is the initial situation. Then we want to find an intermediary finite dimensional algebra $C_1$ such that

$$T[A] \supseteq T[C_1] \not\supseteq T[C].$$

Then we want to find a finite dimensional algebra $C_2$ such that

$$T[A] \supseteq T[C_2] \not\supseteq T[C_1]$$

and so on.

If we shall follow this way then two problems arise.

1) How can one find those intermediary algebras?
2) Does every ascending chain of ideals of identities of finite dimensional algebras terminate?

We shall not discuss the first problem because it is the most technical part of the proof. We construct those algebras using the so-called trace identities.

The solution of the second problem is also technical. The main idea is to translate structure properties of a finite dimensional algebra into the language of identities.

We assume for simplicity that the basic field is algebraically closed. Consider a finite dimensional algebra $C$ with a unit. The algebra $C$ may be represented in the form

$$C = P + \text{Rad } C$$
where \( P \) is the semisimple part of \( C \). The algebra \( P \) is a direct sum of full matrix algebras \( M_{n_i}(F) \). We define the following parameters of the algebra \( C \):

\[
\alpha(C) = \sum n_i \\
\beta(C) = \dim_F P \\
\gamma(C) = \text{the index of nilpotency of } \text{Rad } C.
\]

The triple

\[
t(C) = (\alpha(C), \beta(C), \gamma(C))
\]

is called the type of the algebra \( C \). We order the types lexicographically.

How do we translate the properties of the algebra \( C \) into the language of identities? For example, the property \( \alpha(C) = n \) is translated into the following identity

\[
\sum_{\sigma_1, \ldots, \sigma_k \in S(n+1)} (-1)^{n_1} \cdots (-1)^{n_k} x_1^{\sigma_1(1)} \cdots x_k^{(n+1)} + \cdots x^n (n+1) + \cdots = 0
\]

(2)

where \( k = \gamma(C) \).

If \( T[C'] \supseteq T[C] \) for another finite dimensional algebra \( C' \), then it is proved that there exist finite dimensional algebras \( C_1, \ldots, C_m \) such that

\[
\alpha(C_i) \leq \alpha(C), \quad T[C'] = T \left( \bigoplus_{i=1}^m C_i \right).
\]

Other identities translate some other properties. Finally we can prove the following:

**Proposition.** Let \( C, C' \) be finite dimensional algebras over an infinite field, \( T[C'] \supseteq T[C] \). Then there exist finite dimensional algebras \( C_1, \ldots, C_m \) such that

\[
T \left( \bigoplus_{i=1}^m C_i \right) = T[C'].
\]

This proposition is principal for the solution of the second problem.

### 2. Infinitely Generated PI-Algebras

The methods of studying the identities of infinitely generated algebras highly depend on the characteristic of the basic field.

#### 2.1 Algebras over Field of Zero Characteristic

**Definition 6.** Let \( A \) be an associative algebra, \( A_0, A_1 \) subspaces of \( A \) such that

\[
A = A_0 \oplus A_1 \\
A_0 A_0, A_1 A_1 \subseteq A_0; \quad A_0 A_1, A_1 A_0 \subseteq A_1.
\]
Then the algebra $A = A_0 \oplus A_1$ is said to be the $\mathbb{Z}_2$-graded algebra or superalgebra graded by $A_0, A_1$.

Consider the Grassmann algebra $G$ generated by elements $e_1, e_2, e_3, \ldots$ satisfying the relations $e_ie_j = -e_je_i$ for every $i, j$. Let $G_0$ be the subspace of $G$ generated by monomials of even degree, $G_1$ the subspace generated by monomials of odd degree. Then $G = G_0 \oplus G_1$ is the Grassmann superalgebra.

**Definition 7.** The subalgebra $G(A) = G_0 \otimes_F A_0 + G_1 \otimes A_1$ of the algebra $G \otimes_F A$ is called the Grassman Hull of the superalgebra $A = A_0 \oplus A_1$.

In 1981 the author proved that any T-ideal equals the ideal of identities of the Grassmann Hull of some finitely generated PI-superalgebra. This theorem has reduced the studying of identities of infinitely generated PI-algebras to the studying of graded identities of finitely generated PI-superalgebras. We recall that the characteristic of the basic field equals zero. Theorem 1 formulated above is valid also in the graded case ([3]). Hence we obtain the following main result about the identities of PI-algebras over the field of zero characteristic:

**Theorem 2 [3].** For any PI-algebra $A$ there exists a finite dimensional superalgebra $C$ such that

$$T[A] = T[G(C)].$$

Theorem 2 has an interesting corollary:

**Corollary.** A relatively free PI-algebra can be embedded into the matrix algebra of finite order over an algebra satisfying the identity $[[x, y], z] = 0 ([x, y] = xy - yx)$.

### 2.2 Specht's Problem

We want to classify associative PI-algebras in the language of identities. Theorem 2 is a satisfactory theorem of classification for algebras of zero characteristic. In the general case we do not have any hypothesis about the classification.

**Definition 8.** The minimal set of identities of an algebra $A$ which implies all the other identities of $A$ is said to be the base of identities of the algebra $A$.

In 1950 W. Specht formulated the following problem [5]: Has any associative algebra of zero characteristic a finite base of identities?

The problem of finite base may be formulated not only for algebras of zero characteristic but also for arbitrary algebras, rings, groups and so on. The problem of finite base may be considered as a strict formulation for problem of classification (if there is no more satisfactory hypothesis).

In 1986 the author [6] solved Specht's Problem positively:

**Theorem 3.** Any associative algebra over a field of zero characteristic has a finite base of identities.
Theorem 3 has other formulations:
1) Any T-ideal is finitely generated as a T-ideal.
2) The set of T-ideals satisfies the ascending chain condition.

In the case of non-zero characteristic the problem of finite base is open, but Theorem 1 yields the positive solution of local Specht's problem for algebras over an infinite field.

**Theorem 4.** Any T-ideal of a finitely generated free algebra is generated by a finite set of polynomials (as a T-ideal).

2.3 Identities of Algebras over a Field of Characteristic $p$

In the case of characteristic $p$ Theorem 2 is not true. Moreover the author has no classification hypothesis. The structure of identities of algebras over fields of characteristic $p$ is not clear and there are no strong results on this theme.

In 1981 I.B. Volichenko formulated the strange problem: Does any PI-algebra over a field of non-zero characteristic satisfy the standard identity

$$\sum_{\sigma \in \mathcal{S}(n)} (-1)^{\varepsilon} x_{\sigma(1)} \cdots x_{\sigma(n)} = 0$$

for some $n$?

It is not difficult to prove that Volichenko's hypothesis is true if and only if any PI-algebra of non-zero characteristic satisfies the symmetrized standard identity

$$\sum_{\sigma \in \mathcal{S}(n)} x_{\sigma(1)} \cdots x_{\sigma(n)} = 0$$

for some $n$.

Indeed, let $A$ be any PI-algebra over a field $F$ and $G$ a Grassmann algebra over $F$. It is well-known that the algebra $A \otimes_F G$ is a PI-algebra. If $A \otimes_F G$ satisfies the standard identity (symmetrized standard identity) of some degree then it is easy to see that the algebra $A$ satisfies the symmetrized standard identity (standard identity) of the same degree.

Note that the symmetrized standard identity of $n$-th degree is a full linearization of the identity $x^n = 0$.

The author has solved these problems recently.

**Theorem 5.** Any PI-algebra over a field of non-zero characteristic satisfies the standard identity and the symmetrized standard identity of some degree.

**Proof.** First of all we remark that it is enough to prove the theorem for algebras with unit over algebraically closed field.

Let $\Gamma$ be the T-ideal of identities of the given algebra with unit $\Gamma \subseteq F \langle X \rangle$, $X = \{ x_1, x_2, \ldots \}$. Consider the finitely generated PI-algebras $F_k/\Gamma \cap F_k$, where $F_k = F \langle x_1, \ldots, x_k \rangle$. By Theorem 1

$$T[F_k/\Gamma \cap F_k] = T[A_k]$$

for some finite dimensional algebra $A_k$. 
The primitive images of the algebra $F\langle X \rangle/\Gamma$ are full matrix algebras over the basic field $F$. The maximal order of these matrix algebras is called the complexity of the $T$-ideal $\Gamma$.

We represent the algebra $A_k$ in the form

$$A_k = P_k + \text{Rad} A_k,$$

where $P_k$ is the semisimple part of $A_k$. The algebra $P_k$ may be represented in the form

$$P_k = e_0 P_k + e_1 P_k + \cdots + e_s P_k,$$

where $e_0, e_1, \ldots, e_s$ are orthogonal idempotents, $e_i P_k = M_q(F)$, $i > 0$ ($q$ the complexity of $\Gamma$); $e_0 P_k$ a direct sum of other matrix algebras $M_n(F)$, $n < q$.

We prove the theorem by induction. The base of induction ($q = 1$) and the inductive step will be proved simultaneously. Consider the ideal $I_k$ of the algebra $A_k$ generated by all mixed elements $e_i a e_j$, $i \neq j$, $a \in A_k$. Note that it is sufficient to prove that the algebras $I_k$ satisfy the symmetrized standard identity of some degree $n = n(\Gamma)$.

Indeed, if $q = 1$ we put $k = 4n$. The algebra $A_k/I_k$ satisfies the identity

$$[x, y, \ldots, y]_{p^m} = 0$$

for some $m$. Therefore the algebra $A_k$ satisfies the identities

$$(S_n^+(z_1[x_1, y_1, \ldots, y_1]t_1, \ldots, z_n[x_n, y_n, \ldots, y_n]t_n) = 0$$

($S_n^+$ is the symmetrized standard polynomial), where $z_i, t_j \in X \cup \{1\}$. Since the identities (4) contain at most $k$ variables, these identities are valid modulo $\Gamma$. Hence we obtain that the algebra $F\langle X \rangle/\Gamma$ is an extension of some algebra satisfying (3) by an algebra satisfying the symmetrized standard identity of degree $n$. It is easy to verify that the full linearization of (3) is the symmetrized standard identity of degree $p^m$, hence the algebra $F\langle X \rangle/\Gamma$ satisfies the symmetrized standard identity of degree $p^m \cdot n$.

If $q > 1$ then we may assume that the theorem is proved in the case when the complexity is less than $q$.

Let $B$ be an algebra such that $T[M_q(B)] \supseteq \Gamma$. Since $q$ is the complexity of $\Gamma$ the complexity of the $T$-ideal $T[B]$ is equal to 1. Hence by the inductive hypothesis the algebra $B$ must satisfy the symmetrized standard identity of some degree. It is easy to prove that then the algebra $M_q(B)$ must satisfy the symmetrized standard identity of some degree $m = m(\Gamma)$.

Let $h(x_1, \ldots, x_r)$ be a polynomial such that for all $s$ the algebra $M_q(F)$ does not satisfy the identity $h^s = 0$, but the algebra $M_{q-1}(F)$ satisfies the identity $h^s = 0$ for some $s$.

We put $k = (m + r + 3)n$. Consider the algebra $A_k/I_k$. This algebra is a direct sum of algebras of the type $M_d(B)$, where $B$ is a local algebra, $d \leq q$. Therefore the algebra $A_k/I_k$ must satisfy the identities

$$f(x_1, \ldots, x_m, y_1, \ldots, y_r, u, v, w) = uS_m^+(x_1, \ldots, x_m)v(h(y_1, \ldots, y_r))w = 0$$

(5)
for some \( m, t \), where \( u, v, w \in X \cup \{1\} \). Hence we obtain that the algebra \( A_k \) satisfies the identities

\[
S^+_n(f^{(1)}, \ldots, f^{(0)}) = 0,
\]

where \( f^{(0)} = f(x_1^{(0)}, \ldots, x_m^{(0)}, y_1^{(0)}, \ldots, y_r^{(0)}, u^{(0)}, v^{(0)}, w^{(0)}, u^{(i)}, v^{(i)}, w^{(i)} \in X \cup \{1\} \). Since the identities (6) contain at most \( k \) variables these identities are valid modulo \( \Gamma \). Therefore the algebra \( F(X)/\Gamma \) is an extension of some algebra satisfying (5) by an algebra satisfying the symmetrized standard identity of degree \( n \). By the inductive hypothesis the identity \( h^* = 0 \) implies the symmetrized standard identity of some degree \( n_0 \). Hence we obtain that the algebra \( F(X)/\Gamma \) satisfies the symmetrized standard identity of degree \( (n_0 + m) \cdot n \).

Let us prove that the algebra \( I_k \) satisfies the symmetrized standard identity of some degree \( n = n(\Gamma) \).

Let \( h(x_1, \ldots, x_r) \) be a central polynomial for the algebra \( M_4(F) \). Then the algebra \( F(X)/\Gamma \) satisfies all identities of the type

\[
[h(u_1, \ldots, u_r), v]x^{i_1}[h(u_1, \ldots, u_r), v]x^{i_2} \cdots x^{i_{m-1}}[h(u_1, \ldots, u_r), v] = 0.
\]

Since \( T[A_k] \supseteq \Gamma \), the algebra \( A_k \) also satisfies the identities (7). If \( m > s = s(k) \) then we substitute into the identity (7) the following elements of \( A_k \): \( x = 1, v = e_i a_i e_j a_j e_k e_l \), where \( b_i \) are such elements that \( h(b_1, \ldots, b_r) = \sum \alpha_i e_i, \alpha_i \neq \alpha_j, \alpha_i \in F \) (these elements exist because \( h \) is a central polynomial for \( M_4(F) \)). As a result we get the equality \( v^m = 0 \). Hence we obtain

\[
0 = e_i v^m = e_i a_i e_j a_j \cdots e_m a_m e_{m+1}.
\]

It means that we may assume that \( s \leq m \).

If we substitute into (7) \( x = b \in I_k, v = e_i a_i e_j = e_j \), where \( b_i \) are such elements that \( h(b_1, \ldots, b_r) = \alpha_i e_i, \alpha_i \in F \) (these elements exist because \( h \) is a central polynomial for \( M_4(F) \)). As a result we get the equality \( v^m = 0 \). Hence we obtain

\[
0 = e_i v^m = e_i a_i e_j a_j \cdots e_m a_m e_{m+1}.
\]

Linearizing this equality we can see that if we substitute into the polynomial \( S^+_n(x_1, \ldots, x_n) \) a lot of elements of mixed type \( e_i a_j e_j, i \neq j \), then we annihilate this polynomial. Therefore it is sufficient to prove that for all \( i, j, i \neq j \), the algebra \( B_k = e_i A_k e_j A_k e_j \) satisfies the symmetrized standard identity of some degree \( n = n(\Gamma) \).

Put \( i_1 = i_3 = i_5 = \cdots = 0 \) in (7). Substituting into (7) \( v = y + z, y \in e_i A_k e_j \), \( z \in e_i A_k e_j, x \in e_i A_k e_j, h(v_1, \ldots, v_r) = e_i, \) we get the equality

\[
(z - y)^2 x^{j_1} (z - y)^2 x^{j_2} \cdots x^{j_{m-1}} (z - y)^2 = 0
\]

where \( j_d = i_{2d}, t = [(m + 1)/2] \). Multiplying this equality by \( e_i \) from the left side we get the identity

\[
x_{j_0} y z x^{j_1} y z \cdots x^{j_{m-1}} y z x^{j_t} = 0
\]

for all \( y \in e_i A_k e_j, z \in e_i A_k e_j, x \in e_i A_k e_j \).

Substituting \( y = y + y_1 \) into (8) and taking the component of degree 1 by \( y_1 \), we get the identity

\[
x_{j_0} y_1 z x^{j_1} y z \cdots + x_{j_0} y z x^{j_1} y_1 z \cdots + \cdots = 0.
\]
Substitute \( y_1 = xy_1 \) into (9) and subtract the result from (9) where \( j_0 \) is replaced by \( j_0 + 1 \). We get the identity of the type

\[
x^{j_0} y z x^{j_1} (\alpha_1 y_1 z x^{j_2} y z \cdots + \alpha_2 y z x^{j_2} y_1 z \cdots + \cdots) = 0, \tag{10}
\]

\( \alpha_i \in F \). Then we again substitute \( y_1 = xy_1 \) into (10) and subtract the result from (10) where \( j_1 \) is replaced by \( j_1 + 1 \). Continue the described process. Finally we get a non-trivial identity of the type

\[
\sum_{(j) = (j_0, j_1, \ldots, j_t)} \alpha_{(j)} x^{j_0} y z x^{j_1} \cdots y z x^{j_{t-1}} y_1 z x^{j_t} = 0. \tag{11}
\]

We repeat this process again \( t \) times and then repeat this process with respect to \( z \). Finally we get the identity of the type

\[
\sum_{(j) = (j_0, \ldots, j_t)} \beta_{(j)} x^{j_0} y_1 z_i x^{j_1} \cdots x^{j_{t-1}} y_i z_1 x^{j_t} = 0
\]

for all \( x \in e_i A_k e_j, y \in e_i A_k e_j, z \in e_j A_k e_i \). Hence we obtain that the algebra \( B_k \) satisfies the non-trivial identity of the type

\[
\sum_{(j)} \beta_{(j)} x^{j_0} x_1 x^{j_1} \cdots x_i x^{j_i} = 0 \tag{12}
\]

for all \( x, x_i \in B_k \).

So we may assume that the algebra \( F \langle X \rangle / \Gamma \) satisfies the identity (12). Therefore the algebra \( A_k \) satisfies the identity (12). Substituting \( x = a + e_j, a \in B_k, x_1 = e_i b_1 e_j, x_2 = e_j b_2 e_i, x_i \in B_k, i \geq 3 \), we see that the algebra \( B_k \) satisfies the identity of the type (12) where \( t \) is replaced by \( t - 1 \).

References
