On the Restricted Burnside Problem

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In 1902, W. Burnside formulated his famous problems for periodic groups [6]:

The Burnside Problem (also known as the Ordinary Burnside Problem): Is it true that every finitely generated group of bounded exponent is finite?

The General Burnside Problem: Is it true that every finitely generated periodic group is finite?

After many unsuccessful attempts to obtain a proof in the late 30s–early 40s the following weaker version of The Burnside Problem was studied: Is it true that there are only finitely many m-generated finite groups of exponent \( n \)? In other words the question is whether there exists a universal finite m-generated group of exponent \( n \) having all other finite m-generated groups of exponent \( n \) as homomorphic images. Later (thanks to W. Magnus [35]) this question became known as The Restricted Burnside Problem.

In 1964 E. S. Golod gave a negative answer to The General Burnside Problem (cf. [9]). Since then a considerable array of infinitely generated periodic groups was constructed by other authors (cf. Alyoshin [2], Suschansky [44], Grigorchuk [11], Gupta-Sidki [54]).

In 1968 P. S. Novikov and S. I. Adian [39] constructed counter-examples to The Burnside Problem for groups of odd exponents \( n \geq 4381 \) (now for odd exponents \( n \geq 115 \), cf. I. Lysenok [33]). Olshansky's Monsters (cf. [40]) shows how wildly periodic groups may behave.

At the same time there were two major reasons to believe that The Restricted Burnside Problem would have a positive solution. One of these reasons was the reduction theorem obtained by Ph. Hall and G. Higman [14]. Let \( n = p_1^{k_1} \ldots p_r^{k_r} \), where \( p_i \) are distinct prime numbers, \( k_i \geq 1 \), and assume that (a) The Restricted Burnside Problem for groups of exponents \( p_i^{k_i} \) has a positive solution, (b) there are only a finite number of finite simple groups of exponent \( n \), (c) the factor group \( \text{Out}(G) = \text{Aut}(G)/\text{Inn}(G) \) is solvable for any finite simple group of exponent \( n \). Then The Restricted Burnside Problem for groups of exponent \( n \) also has positive solution.

Another reason was the close relation of The Problem to Lie algebras. Suppose \( n = p^k \), where \( p \) is a prime number. Then the finite group \( G \) of exponent \( p^k \) is clearly nilpotent. It is easy to see that it is sufficient to find an upper bound
Consider the lower central series \( G = \gamma_1(G) > \gamma_2(G) > \cdots > \gamma_s(G) = (1), \gamma_{i+1}(G) = (\gamma_i(G), G), 1 \leq i < S \), and the direct sum of abelian groups

\[
L_\gamma(G) = \bigoplus_{i=1}^{s-1} \gamma_i(G)/\gamma_{i+1}(G).
\]

Brackets \([a_1 \gamma_{i+1}(G), b_1 \gamma_{j+1}(G)] = (a_1, b_1) \gamma_{i+j+1}(G)\), where \( a_i \in \gamma_i(G), b_j \in \gamma_j(G) \), is the group commutator, define the structure of a Lie ring on \( L_\gamma(G) \). It is obvious that the Lie ring \( L_\gamma(G) \) has the same class of nilpotency as the group \( G \). If \( G \) is generated by elements \( x_1, \ldots, x_m \) then \( L_\gamma(G) \) is generated by \( x_i \gamma_2(G), 1 \leq i \leq m \).

If \( n = p \) is a prime number then \( L_\gamma(G) \) is an algebra over the field \( \mathbb{Z}_p, |\mathbb{Z}_p| = p \), which satisfies Engel's identity (cf. [35]). Thus The Problem for groups of prime exponent has been reduced to the following problem in Lie algebras: is it true that a Lie algebra over \( \mathbb{Z}_p \), which satisfies Engel's identity \( E_{p-1} \), is locally nilpotent?

The last problem was successfully solved by A. P. Kostrikin [27, 28] who solved in this way The Restricted Burnside Problem for groups of prime exponent.

If \( G \) is a finite group of prime power exponent \( p^k \) then \( L_\gamma(G) \) is no longer an algebra over the field \( \mathbb{Z}_p \), it is an algebra over the ring \( \mathbb{Z}(p^k) \) of residues modulo \( p^k \). That's why along with the lower central series \( \gamma_i(G) \) we shall consider the lower central \( p \)-series of \( G \) (cf. [21], [31, 48, 46]):

\[
G = G_1 \triangleright G_2 \triangleright \cdots ,
\]

where \( G_i \) is the subgroup of \( G \) generated by commutators \((\ldots(x_1, x_2), x_3), \ldots, x_r\), \( r \geq i \), and powers \((\ldots(x_1, x_2), x_3), \ldots, x_r)^{p^r}, r \cdot p^r \geq i \). It is easy to see that

\[
L(G) = \bigoplus_{i \geq 1} G_i/G_{i+1}
\]

is an algebra over \( \mathbb{Z}_p \). Neither the Lie ring \( L_\gamma(G) \) nor \( L(G) \) need necessarily satisfy Engel's identity \( E_{p^k-1} \) (cf. [13, 16]) but

(1) the Lie algebra \( L(G) \) satisfies the linearized Engel's identity \( E_{p^k-1} \), that is, for arbitrary elements \( a_1, \ldots, a_{p^k-1} \in L(G) \) we have

\[
\Sigma \text{ad} (a_{\sigma(1)}) \ldots \text{ad} (a_{\sigma(p^k-1)}) = 0, \quad \sigma \in S_{p^k-1},
\]

(G. Higman, [17]).

(2) for an arbitrary commutator \( q \) on the generators \( x_i G_2, 1 \leq i \leq r \), we have

\[
\text{ad} (q)^{p^k} = 0
\]

(I. N. Sanov, [42]).
Now let us turn to what was happening in associative and Lie nil algebras.

A. G. Kurosch [30]* and independently J. Levitzky (cf. [3]) formulated two problems for nil algebras which were similar to Burnside's problems.

**The General Kurosh-Levitzky Problem:** Is every finitely generated nil algebra nilpotent?

**The (Ordinary) Kurosh-Levitzky Problem:** Is every finitely generated nil algebra of bounded degree nilpotent?

Actually it was a counterexample to the General Kurosh-Levitzky Problem which was constructed in the paper, [9] of E. S. Golod, then this counterexample was used to construct the first counterexample to the General Burnside Problem. Remark that so far it remains the only counterexample to the General Kurosh-Levitzky Problem.

For the (Ordinary) Kurosh-Levitzky Problem we have a quite different situation. Unlike Group Theory it has only positive solutions in all important classes of algebras. To appreciate the impact this problem had on the Ring Theory let us mention that N. Jacobson's Structure Theory of Algebras was stimulated by the Kurosh Problem and I. Kaplansky introduced the concept of a PI-algebra in search of the most general conditions which ensure the positive solution of the Problem. The following result (in its final form) was due to I. Kaplansky [23].

**Theorem.** A finitely generated nil algebra which satisfies a polynomial identity is nilpotent.

In 1956 A. I. Shirshov suggested another purely combinatorial direct approach to the Kurosh-Levitzky Problem.

**Theorem** (A. I. Shirshov [43]). Suppose that an associative algebra $A$ is generated by elements $x_1, \ldots, x_r$ and assume that (1) $A$ satisfies a polynomial identity of degree $n$, (2) every monomial in $\{x_i\}$ of degree $\leq n$ is nilpotent. Then $A$ is nilpotent.

It is very important that the nilpotency assumption here is imposed not on every element of $A$, but only on the monomials in the generators (even on a finite collection of them).

Now let us turn to Lie algebras. It is natural to call an element $a \in L$ nilpotent if the operator $\text{ad}(a)$ is nilpotent. With this definition both Kurosh-Levitzky problems become meaningful for Lie algebras. Moreover, by the results of G. Higman and I. N. Sanov (cf. above) the Lie algebra $L(G)$ of a finite group $G$ of exponent $p^n$ satisfies the assumptions of the Kurosh-Levitzky Problem in the form of A. I. Shirshov (the role of monomials is played by commutators).

In [52, 43] we solved this problem for those Lie algebras which satisfy the linearized Engel's identity $E_n$.

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* Actually Kurosh's Problem concerned algebraic algebras but we consider only the important case of nil algebras.
Theorem 1. Suppose that a Lie algebra \( L \) is generated by elements \( x_1, \ldots, x_r \) and assume that there exist integers \( n \geq 1, m \geq 1 \) such that (1) \( L \) satisfies the linearized Engel's identity \( E_n \), (2) for an arbitrary commutator \( q \) on the generators \( x_i \) we have \( \text{ad}(q)^m = 0 \). Then \( L \) is nilpotent.

Corollary. A Lie ring which satisfies Engel's identity is locally nilpotent.

From Theorem 1 we derive

Theorem 2. The Restricted Burnside Problem has a positive solution for groups of exponent \( p^k \).

By the Reduction Theorem of Ph. Hall and G. Higman, the Restricted Burnside Problem has a positive solution in the class of soluble groups, in particular it has a positive solution for groups of odd exponent (by the celebrated Theorem of W. Feit and J. Thompson [8]) and for groups of exponent \( n = p^aq^b \), \( p, q \) are prime numbers (cf. W. Burnside, [7]).

The announced classification of finite simple groups (cf. [10]) implies that the Restricted Burnside Problem has a positive solution for groups of arbitrary exponent. Now we shall try to explain briefly the idea of the proof of Theorem 1.

In [50] we proved that to prove Theorems 1, 2 it suffices to prove that a restricted (in the sense of N. Jacobson [18]) Lie algebra over an infinite field, which satisfies an Engel's identity, is locally nilpotent. An element \( a \) of a Lie algebra \( L \) is called sandwich if

\[
[[L, a], a] = 0, \quad [[[L, a], L], a] = 0
\]

(cf. A. I. Kostrikin, [27]). In case of odd characteristics the second equality easily follows from the first one, however if \( \text{char} = 2 \) then both conditions are necessary. We call a Lie algebra a sandwich algebra if it is generated by a finite collection of sandwiches. The following theorem is due to A. I. Kostrikin and the author.

**Theorem About Sandwich Algebras** [49]. A sandwich Lie algebra is nilpotent.

This theorem suggests the following plan of attack on Theorem 1 (which has been outlined in [50]). Assume that there exists a nonzero Lie algebra \( L \) over an infinite field \( K \) which satisfies an Engel's identity but isn't locally nilpotent. Then taking the factor-algebra of \( L \) modulo its locally nilpotent radical (cf. [26, 41]) we may assume \( L \) doesn't contain any nonzero locally nilpotent ideals. Suppose we manage to construct a Lie polynomial \( f(x_1, \ldots, x_r) \) such that \( f \) is not identically zero on \( L \) and for arbitrary elements \( a_1, \ldots, a_r \in L \) the value \( f(a_1, \ldots, a_r) \) is a sandwich of \( L \). The \( K \)-linear span of \( f(L) = \{f(a_1, \ldots, a_r) | a_1, \ldots, a_r \in L \} \) is an ideal in \( L \). By the theorem about sandwich algebras the ideal \( Kf(L) \) is locally nilpotent which contradicts our assumption.

However one year of effort didn't bring us a desired sandwich-valued polynomial (its existence a posteriori follows from Theorem 1). Instead in November of 1988 we constructed an even sandwich-valued superpolynomial \( f \), which means
that for a Lie superalgebra $L = L_0 + L_1$ satisfying the superization of $E_n$ every value of $f$ is a sandwich of $L_0$. It turned out to be a good substitute of sandwich-valued polynomials. The sketch of this rather complicated construction appeared in [51]. Unfortunately it worked only for characteristics $\neq 2, 3$.

In January of 1989 we constructed another "generalized" nonzero sandwich-valued polynomials (this time involving "divided powers" of ad-operators), their full linearizations being ordinary polynomials. Every value of such a full linearization is a linear combination of sandwiches. This approach worked for any $p$ (cf. [52, 53]).

Some lengthy computations from the proof (which are really hard to read) may be explained within the framework of Jordan Algebra Theory (cf. [19, 20]). We shall demonstrate the idea for the simpler case $p \neq 2, 3$. The first (less computational) part of the construction of a sandwich-valued polynomial is a construction of a polynomial $f$ such that $f(L) \neq 0$ and for an arbitrary element $a \in f(L)$ we have $\text{ad} (a)^3 = 0$. Choose arbitrary elements $a, b \in f(L)$ and consider the subspaces $\mathcal{D}^+ = L \text{ ad} (a)^2$, $\mathcal{D}^- = L \text{ ad} (b)^2$. Then for an arbitrary element $c \in \mathcal{D}^-$ the operation $x \circ y = [x, c, y]; x, y \in \mathcal{D}^+$, defines the structure of a Jordan algebra on $\mathcal{D}^+$ (cf. [5, 19, 22, 25]). The pair of subspaces $(\mathcal{D}^-, \mathcal{D}^+)$ is a so-called Jordan pair (cf. [32, 38]).

For $p = 2$ or $p = 3$ we define $\mathcal{D}^-$ and $\mathcal{D}^+$ with the divided powers of adjoint-operators and apply Kevin McCrimmon's Theory of Quadratic Jordan Algebras ([20, 36, 37]).

For odd $p$ we managed to translate Jordan arguments into the language of elementary computations in [52]. For $p = 2$ this substitute didn't work so Jordan Pairs and Algebras played an important role in our paper [53]. However, recently M. Vaughan-Lee succeeded in getting rid of Jordan Algebra Theory even in the case $p = 2$.

Not much is known about the upper bound for classes of nilpotency of $r$-generated finite groups of exponent $p$ (let alone the exponent $p^k, k > 1$). S. I. Adian and N. N. Repin [1] proved that it grows at least exponentially with respect to $p$. For comparison let us mention the recent result of A. Belov which asserts that there exists a constant $\alpha$ such that an arbitrary $r$-generated associative ring which satisfies the identity $x^n = 0$ is nilpotent of degree $\leq r^{2n}$.

**Conjecture.** There exists a constant $\alpha$ such that an arbitrary $r$-generated finite group of exponent $p$ is nilpotent of class $\leq r \cdot \alpha^p$.

**Residually Finite Groups and Compact Groups**

The following generalization of Theorem 1 solves the Kurosh-Levitzky Problem (in Shirshov's form) in the class of Lie PI-algebras.

**Theorem 3.** Suppose that a Lie algebra $L$ is generated by a finite subset $X \subseteq L$, $|X| = m$ and assume that

1. $L$ satisfies a polynomial identity of degree $n$. 

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(2) 

for an arbitrary commutator \( q \) on \( X \) of weight \( \leq h(m, n) \) the operator \( \text{ad}(q) \) is nilpotent.

Then \( L \) is nilpotent.

This theorem has some applications to the General Burnside Problem in group varieties and to compact groups. It is well known that there are counterexamples to the General Burnside Problem even among residually finite \( p \)-groups (such are the counterexamples of E. S. Golod and R. I. Grigorchuk). However Theorem 3 implies

**Theorem 4.** A residually finite \( p \)-group which satisfies a nontrivial group identity is locally finite.

Apparently this theorem can be generalized from \( p \)-groups to periodic groups in the spirit of the theorem of P. Hall and G. Higman [14].

The assumption of a nontrivial identity can be further weakened to the "infinitesimal" assumption that the adjoint Lie algebra \( L(G) \) is PI. The last assertion in its turn is related to the General Burnside Problem for compact groups. V. P. Platonov conjectured that periodic compact (Hausdorff) groups are locally finite. J. S. Wilson [47] proved that (under the assumption that there are finitely many simple sporadic groups) it suffices to prove the conjecture for pro-\( p \)-groups. That's what is done in the following theorem.

**Theorem 5.** Every periodic pro-\( p \)-group is locally finite.

Indeed, let \( G \) be a periodic pro-\( p \)-group. Consider the closed subsets \( G_{(n)} = \{g \in G | g^{p^n} = 1\}, G = U G_{(n)} \). By Baire’s Category theorem one of the subsets \( G_{(n)} \) contains some neighborhood, that is \( G_{(n)} \supseteq gH \), where \( H \) is a normal subgroup of \( G \) of finite index. Then we show that every finitely generated subgroup of \( H \) which is invariant under conjugation by \( g \), satisfies an "infinitesimal" identity.

From Theorem 5 combined with [47] and with what is known about locally finite groups [15, 24] there follows

**Theorem 6.** Every infinite compact group contains an infinite abelian subgroup.

Remark as far as Theorem 6 is concerned the reduction to pro-\( p \)-groups in [47] didn’t use the classification of finite simple groups.

**References**


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