Rational Homotopy Theory and Deformation Problems from Algebraic Geometry

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This paper is a description of research I have been doing over the last four years, applying some of the methods and ideas of rational homotopy theory as developed by Chen, Quillen and Sullivan, to deformations of flat and holomorphic bundles, complex manifolds and isolated singularities. My work is based on the fundamental observation of Pierre Deligne [D] that "in characteristic zero, a deformation problem is controlled by a differential graded Lie algebra, with quasi-isomorphic differential graded Lie algebras giving the same deformation theory." I would like to thank Pierre Deligne for providing me with this insight and Bill Goldman who was my collaborator in developing much of what follows. I would also like to thank Ragnar Buchweitz, Kevin Corlette, Steve Halperin, Jack Lee, Madhav Nori and Mike Schlessinger for helpful conversations. The interested reader will find details in [GM1, GM2, BM] and [M].

We begin by recalling that Sullivan showed how to recover explicitly the rational homotopy type of a simply-connected manifold $M$ by replacing the de Rham algebra $\mathcal{A}^*(M)$ on $M$ by a minimal free differential graded algebra quasi-isomorphic to it having finite dimensional cochain groups and decomposable differential. We recall that two differential graded algebras $A$ and $B$ are quasi-isomorphic if there is a chain of homomorphisms

$$A = A_0 \rightarrow A_1 \leftarrow A_2 \cdots \rightarrow A_n = B$$

all of which induce isomorphisms of cohomology. The rational homotopy type of $M$ can be calculated from any differential graded algebra quasi-isomorphic to $\mathcal{A}^*(M)$.

We will make use of one concept from the now well-developed homotopy theory of differential graded algebras.

**Definition.** A differential graded algebra $A$ is formal if it is quasi-isomorphic to a differential graded algebra $B$ with zero differential. The underlying algebra $B$ is then necessarily isomorphic to the cohomology algebra of $A$.

We now recall the celebrated theorem of [DGMS].

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Theorem. The de Rham algebra $\mathcal{A}^*(M)$ of a compact Kähler manifold $M$ is formal.

The theorem is a consequence of the following considerations. We use the complex structure on $M$ to decompose the exterior differential $d$ on $M$ in the usual way, $d = \partial + \overline{\partial}$. Then we have the following quasi-isomorphism

$$(\mathcal{A}^*(M), d) \leftarrow (\ker \partial, \overline{\partial}) \rightarrow \left( \frac{\ker \partial}{\text{im} \partial}, 0 \right)$$

The fact that $\overline{\partial}$ induces the zero differential on $\mathcal{A}^1$ is an immediate consequence of the "$\partial\overline{\partial}$-lemma" of Kähler geometry, [DGMS].

To carry Sullivan's ideas over to deformation theory, we start with a differential graded Lie algebra over a field $k$ (we will consider only those algebras with finite dimensional first cohomology groups). Given such an algebra $L$ we choose a complement $C^1(L)$ to the 1-coboundaries $B^1(L) \subset L^1$. We define a functor $A \rightarrow Y_L(A)$ on the category of Artin local $k$-algebras by

$$Y_L(A) = \{ \eta \in C^1(L) \otimes m : d\eta + \frac{1}{2}[\eta, \eta] = 0 \}.$$ 

Here $m$ is the maximal ideal of the Artin local $k$-algebra $A$. It is evident that the functor $Y_L$ satisfies the hypotheses of Theorem 2.11 of [Sc1] and is consequently pro-representable by a complete local $k$-algebra $R_L$ (we will see later as a consequence of our main theorem that the isomorphism class of $R_L$ does not depend on the complement $C^1(L)$).

One can apply the above construction to the following geometric situations:

(i) The twisted de Rham algebra with coefficients in the flat Lie algebra bundle $\text{ad} P$ associated to a flat principal $G$-bundle $P$ over a compact manifold $M$;

(ii) the Kodaira-Spencer algebra $(\otimes \mathcal{A}^0, (M, T^{1,0}(M)), \overline{\partial})$ associated to a complex manifold $M$;

(iii) the tangent complex $T$ associated to the germ $(V, x)$ of an isolated singularity in $\mathbb{C}^n$.

It is then reasonably clear (and proved in [GM2]) from the above construction that in cases (i) and (ii) the algebra $R_L$ is the completion of the analytic local ring $\mathcal{O}_M$ associated to the versal deformation space as constructed by Kuranishi of the given flat connection on $P$ (resp. complex structure on $M$). In case (iii), it is proved in [BM] following ideas in [SS] that the ring $R_T$ is isomorphic to the completion of the analytic local ring of the versal deformation space of the isolated singularity $(V, x)$. We note two other important cases of differential graded Lie algebras $L$ over a field $k$ in which $R_L$ is the completion of an analytic local $k$-algebra.

In case $L$ has zero differential, then $R_L$ is the completion of the analytic local $k$-algebra associated to the germ $(\mathcal{O}_L, 0)$ where

$$\mathcal{O}_L = \{ \eta \in L^1 : [\eta, \eta] = 0 \}.$$ 

We note that $\mathcal{O}_L$ is a quadratic cone canonically associated to $L$. In case $L^1$ is finite dimensional, then $R_L$ is the completion of the analytic local $k$-algebra of the germ $(Y, 0)$ where
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\[ Y = \{ \eta \in C^1 : d\eta + \frac{1}{2}[\eta, \eta] = 0 \}. \]

Henceforth we will use \( \mathcal{X}_L \) to denote the corresponding analytic germ.

**Definition.** If \((X, x)\) is an analytic germ which parameterizes a versal family for a deformation theory and \( L \) is a differential graded Lie algebra such that \( R_L \approx \hat{\mathcal{H}}_{X,x} \) then we will say that \( L \) is a controlling differential graded Lie algebra for that deformation theory.

We have the following theorem (Theorem 4.1 of [GM2]).

**The Comparison Theorem.** Suppose \( f : L_1 \to L_2 \) is a homomorphism of differential graded Lie algebras such that \( f \) induces an isomorphism on first cohomology and an injection on second cohomology. Then \( R_{L_1} \) and \( R_{L_2} \) are isomorphic.

**Corollary.** The isomorphism class of \( R_L \) does not depend on the choice of \( C^1(L) \).

**Corollary.** If \( L_1 \) and \( L_2 \) are quasi-isomorphic, then \( R_{L_1} \) and \( R_{L_2} \) are isomorphic.

**Remarks.** We do not require \( f \) to carry the complement \( C^1(L_1) \) into \( C^1(L_2) \). In case \( R_{L_1} \) and \( R_{L_2} \) are the complete local \( k \)-algebras associated to analytic germs \((X_1, x_1)\) and \((X_2, x_2)\), it follows from [A] that \((X_1, x_1)\) and \((X_2, x_2)\) are isomorphic.

If \( L \) is quasi-isomorphic to a differential graded Lie algebra with zero differential (one says \( L \) is formal), then it follows that

\[ \mathcal{H}_L = \{ \eta \in H^1(L) : [\eta, \eta] = 0 \}. \]

Somewhat surprisingly this frequently happens. Carlos Simpson [S] has shown that the twisted de Rham algebra is formal if \( M \) is Kähler, the structure group \( G \) of the underlying principal bundle is linear and the monodromy representation \( \varphi : \pi_1(M) \to G \) is completely reducible. Using Simpson’s Theorem, the above general results and some standard results relating representations of \( \pi_1(M) \) and flat connections we obtain the following theorem (recall that if \( G \) is a linear algebraic group over \( \mathbb{R} \) (or \( \mathbb{C} \)) then the space of representations \( \text{Hom}(\pi_1(M), G) \) is an affine scheme over \( \mathbb{R} \) (or \( \mathbb{C} \))). The proof of this theorem is in [GM1].

**Theorem.** Let \( G \) be a linear algebraic group over \( \mathbb{R} \) (or \( \mathbb{C} \)) and \( M \) be a compact Kähler manifold. Let \( \varphi : \pi_1(M) \to G \) be a completely reducible representation. Then the analytic local ring of \( \text{Hom}(\pi_1(M), G) \) at \( \varphi \) is isomorphic to that of the quadratic germ \((2,0)\) where

\[ 2 = \{ u \in Z^1(\pi_1(M), g) : [u, u] = 0 \text{ in } H^2(\pi_1(M), g) \}. \]

Here \( Z^1(\pi_1(M), g) \) is the space of Eilenberg-MacLane 1-cocycles with values in the Lie algebra \( g \) of \( G \) and \([u, u]\) denotes the product obtained by combining the cup-product on group cochains with the bracket on \( g \).
The proofs of the above results are too long to be given here; however, to emphasize the analogy with the theorem of [DGMS] referred to earlier, we give the original proof of [GM1] of the formality of the twisted de Rham algebra in case G is compact. We may decompose the exterior covariant differential \( d_V \) by type as \( d_V = \partial_V + \bar{\partial}_V \) (here \( V \) is the covariant derivative operator on sections of \( \text{ad} P \otimes \mathbb{C} \) associated to the flat connection on \( P \)). We then have the quasi-isomorphism

\[
(A'\star(M, \text{ad} P_\mathbb{C}), d_V) \leftrightarrow (\ker \partial_V, \bar{\partial}_V) \rightarrow \left( \frac{\ker \partial_V}{\text{im} \partial_V}, 0 \right).
\]

Once again the induced differential on \( \frac{\ker \partial_V}{\text{im} \partial_V} \) is zero by a \( \partial_V \bar{\partial}_V \)-lemma, see [GM1].

There is also an interesting formality result for the Kodaira-Spencer algebra.

**Theorem.** Let \( M^n \) be a compact Kähler manifold admitting a nowhere zero top degree holomorphic form. Then the Kodaira-Spencer algebra of \( M \) is formal. Moreover, the cup-square from \( H^1(M, T^{1,0}(M)) \) to \( H^2(M, T^{1,0}(M)) \) is zero.

**Corollary.** The Kuranishi space (versal deformation space) of \( M \) is \( H^1(M, T^{1,0}(M)) \).

The corollary is Bogomolov's Theorem. Our proof of the above theorem is a reinterpretation of proofs of Tian [Ti] and Todorov [To]. We let \( \omega \) be the nowhere zero top degree holomorphic form on \( M \). We then obtain an isomorphism of complexes \( \Phi : (A^{0,*}(M, T^{1,0}(M)), \bar{\partial}) \rightarrow (A^{n-1,*}(M), \bar{\partial}) \) given by

\[
\Phi(\eta) = \iota_{\eta} \omega.
\]

Here \( \iota_{\eta} \omega \) denotes the contraction [FN] of the scalar form \( \omega \) by the vector form \( \eta \). We use \( \Phi \) to transfer the graded Lie bracket from the Kodaira-Spencer algebra to \( \bigoplus_{q=0}^{n} A^{n-1,q}(M) \).

**The Tian-Todorov Lemma.** Suppose \( \eta_1 \in A^{n-1,q_1}(M) \) and \( \eta_2 \in A^{n-1,q_2}(M) \) are both \( \partial \)-closed differential forms. Then the (transported) bracket \([\eta_1, \eta_2]\) is \( \partial \)-exact.

**Remark.** This lemma is the analogue of the lemma in symplectic geometry that the bracket of two symplectic vector fields corresponds to an exact 1-form. It may be proved in the same way using the formalism of vector-valued forms of [FN].

Once the Tian-Todorov Lemma is proved, the formality of the Kodaira-Spencer algebra follows from the now familiar diagram

\[
(A^{n-1,*}(M), \bar{\partial}) \leftrightarrow (\ker \partial, \bar{\partial}) \rightarrow \left( \frac{\ker \partial}{\text{im} \partial}, 0 \right).
\]

Here \( \ker \partial \) denotes \( \ker (\partial : A^{n-1,*}(M) \rightarrow A^{n,*}(M)) \) and \( \text{im} \partial \) denotes \( \text{im} (\partial : A^{n-2,*}(M) \rightarrow A^{n-1,*}(M)) \). For details and the vanishing of the cup square the reader is referred to [GM2].
A class of examples which are easily analyzed is the class of compact complex parallelizable nilmanifolds. Let \( M = \Gamma \backslash N \), \( N \) nilpotent complex with Lie algebra \( \mathfrak{n} \) defined over \( \mathbb{R} \) and \( \Gamma \) a cocompact lattice. Let \( L \) be the Kodaira-Spencer algebra and \( \overline{L} \subset L \) the image of the left \( N \)-invariants. The inclusion \( \overline{L} \to L \) is a quasi-isomorphism so \( \mathcal{K}_L \cong \mathcal{K}_{\overline{L}} \). It is easy to see that

\[
\mathcal{K}_{\overline{L}} = \text{End}_{\text{alg}}(\mathfrak{n})
\]

the (germ at 0 of the) affine variety of Lie algebra endomorphisms of \( \mathfrak{n} \). If we describe \( \mathfrak{n} \) by generators and relations we can produce a very large number of germs that are Kuranishi spaces of complex manifolds. For example, let \( \mathfrak{n} \) be the free Lie algebra on two generators \( X \) and \( Y \) subject to the relations

(i) all \((n+1)\)-fold commutators = 0
(ii) \( \text{ad}^{n-1} X(Y) = \text{ad}^{n-1} Y(X) \).

Then

\[
\mathcal{K}_{\overline{L}} = \{(X', Y') : \text{for some } X', Y' \text{ satisfying (ii)}\}
\]

so the Kuranishi space of \( M \) is a homogeneous cone of degree \( n \), see [GM2] for more details.

Remark. These examples provide realizations of all the obstructions to integrating an infinitesimal deformation of the complex structure of a complex manifold \( M \).

We observe that \( \overline{L} \) controls the deformation theory of locally left-invariant (i.e., descended from left-invariant complex structure on \( N \)) structures on \( M \). Since \( \overline{L} \to L \) is a quasi-isomorphism it follows that the two deformation theories are the same, that is, every complex structure on \( M \) sufficiently close to the locally bi-invariant one is locally left-invariant. The rest of this paper outlines deeper examples of such a comparison of two deformation theories.

We now describe applications of the above ideas to deformation of isolated singularities. We first summarize the fundamental results of Kuranishi in [K2]. Let \( V \) be an analytic subvariety of \( \mathbb{C}^N \) with a normal isolated singularity at the origin. Let \( M \) be a link of \( V \) (the intersection of \( V \) with a small sphere centered at the origin). Then \( M \) has an induced CR-structure. Let \( T^{1,0}(M) \) be the \((1,0)\) subspace of the complexified horizontal subspace \( H \otimes \mathbb{C} \subset T(M) \otimes \mathbb{C} \). Choose a complement \( F \) to \( H \) in \( T(M) \). The map \( \tau \) of [K2], (8), gives

\[
E = T^{1,0}(M) \oplus (F \otimes \mathbb{C})
\]

the structure of a holomorphic vector bundle over \( M \). Then \((\oplus \mathcal{A}^0, (M, E), \overline{b})\) is a complex, and Kuranishi used it to construct a finite-dimensional family \((\mathcal{K}_M, 0)\) of integrable CR-structures on \( M \) which is a versal deformation of the given CR-structure modulo a relation (coarser than isomorphism) designed to account for the above choice of sphere. However, he did not give \( \mathcal{K}_M \) a complex analytic structure nor did he relate \( \mathcal{K}_M \) to the versal deformation space of \((V', 0)\). The first problem was solved for the case \( \dim V \geq 4 \) by Miyajima in [Mi1] completing earlier work of Akahori [Ak1]. In the rest of this paper we will
show how the Comparison Theorem can be used to identify \( \mathcal{H}_M \) with the versal deformation space of \((V,0)\). The following result is proved in [BM]. It was proved independently by Miyajima in [Mi2] using results of Fujiki [F].

**Theorem.** Suppose \((V,0)\) is normal and satisfies

1. \( \dim V \geq 4; \)
2. \( \text{depth}_{(0)} V \geq 3. \)

Then the base space of the versal deformation of \((V,0)\) is isomorphic to \( \mathcal{H}_M \).

**Remarks.** The assumption (2) is equivalent to the assumption that the Kohn-Rossi cohomology group \( H^1(M, \mathcal{D}) \) vanishes by [Y]. If we do not assume (2), it can be shown that the base space for the versal deformation of \((V,0)\) is isomorphic to a closed subgerm of \( \mathcal{H}_M \). In [BM] we give a family of examples such that \((V,0)\) is normal but the deformation space of \((V,0)\) is a proper subgerm of \( \mathcal{H}_M \).

We will now prove the above theorem by applying the Comparison Theorem many times.

Let \( T \) be the tangent complex associated to \((V,0)\), see [B], [Sc3] or [P]. Then as stated above, \( T \) is a differential graded Lie algebra that controls the deformation theory of \((V,0)\). We recall that this means that there is an isomorphism

\[ R_T \cong \mathcal{D}_{X,0} \]

where \((X,0)\) is the analytic germ parametrizing the versal deformation of \((V,0)\).

Choose any Stein representative \( V \) of \((V,0)\) and let \( L \) be the Kodaira-Spencer algebra of \( U = V - \{0\} \). In [BM] we prove, following [Sc2], that under the assumption \( \text{depth}_{(0)} V \geq 3 \) we have

\[ R_T \cong R_L. \]

It remains to compare the deformation theory of \( U \) with that of \( M \).

We now consider the image of \( L \) under the restriction map from \( U \) to \( M \). Unfortunately brackets in \( L \) do not behave well under this map and consequently it is necessary to replace \( L \) by a subalgebra \( L_{\text{tan}} \) of forms whose restrictions to \( M \) take values tangent to \( M \). The definition of the algebra \( L_{\text{tan}} \) is somewhat involved. We may assume that \( r \) has no critical points on \( U \) and note that \( \omega = \frac{1}{2} \partial \partial \bar{\partial}(r^2) \) is the restriction of the Kähler form of \( \mathbb{C}^n \). We consider the sub-graded vector space \( D^* \) of \( L \) defined by

\[ D^0 = \{ Z \in L^0 : i \bar{\partial} Z \partial r|_M = 0 \} \]

and for \( i \geq 1 \)

\[ D^i = \{ \mu \in L^i : i \mu \partial r|_M = 0, \quad i \mu \omega|_M = 0 \}. \]

It is proved in [Ak2] that \( (D^*, \bar{\partial}) \) is a complex and it is easily seen that

\[ D^+ = \bigoplus_{i \geq 1} D^i \]
is closed under the Frölicher-Nijenhuis bracket [FN]. Furthermore, it is a result of Akahori [Ak2] that the inclusion $D^* \to L^*$ is a quasi-isomorphism of complexes. Thus if we choose a complement $C^1$ to $\overline{\partial}D^0$ in $D^1$ and define

$$L_{\tan} = C^1 \oplus \bigoplus_{i \geq 2} D^i,$$

we find that $L_{\tan}$ is a differential graded Lie algebra and that the inclusion $L_{\tan} \to L$ induces an isomorphism of cohomology in degree greater than or equal to one. Thus by the Comparison Theorem we have (for any choice of $C^1$)

$$R_{L_{\tan}} \approx R_L.$$

We now identify the image of $j^*$. To do this we need the CR-analogue of $L_{\tan}$ which was constructed by Akahori in [Ak1] and was the basis of the proof of the Akahori-Miyajima Theorem referred to above. We define $D^*$ to be the following sub-graded vector space of Kuranishi’s complex $(\mathfrak{A}^*(M, E), \partial_b)$. Let $\theta$ be a contact form on $M$ compatible with $H$. Then we define

$$D^0 = \{ Z \in \mathfrak{A}^0(M, E) : i^*_{\partial^b} \theta = 0 \}$$

and for $i \geq 1$,

$$D^i = \{ \mu \in \mathfrak{A}^i(M, E) : i^*_{\partial^b} \mu \theta = 0, \quad i^*_{\partial^b} d \theta = 0 \}.$$ 

We then choose a complement $C^i$ to $\overline{\partial}_b D^0$ in $D^1$ and define

$$L = C^1 \oplus \bigoplus_{i \geq 2} D^i.$$ 

We also define

$$D^+ = \bigoplus_{i \geq 1} D^i.$$ 

Now the restriction map $j^*$ followed by the canonical map $\tau : T^{1,0}(U)|M \to E$ of [K2] gives a homomorphism of complexes

$$j^+ : D^+ \to D^+.$$ 

Let $I^+ = \ker j^+$. 

**Lemma.** $I^+$ is an ideal in $D^+$. 

As a consequence of this lemma we find that $D^+$ carries the structure of a differential graded Lie algebra such that $j^+$ is a homomorphism. Since $j^*$ is surjective we may choose the complement $C^1$ so that it is carried into $C^1$ by $j^+$. We obtain a homomorphism of differential graded Lie algebras

$$j^+ : L_{\tan} \to L.$$ 

In [Ak1], Akahori proved that the inclusion of $L$ into Kuranishi’s complex induced an isomorphism of cohomology of degree one or greater. In [Ak1] and
[Mi1], Akahori and Miyajima proved that Kuranishi's family $\mathcal{X}_M$ was isomorphic to the analytic subvariety $\mathcal{X}_L \subset H^1(\mathcal{L})$ obtained by applying the well-known construction of [K1] to $\mathcal{L}$ with a suitable choice of complement $\mathcal{C}^1$, provided $\dim V \geq 4$. In particular, it is immediate that

$$R^*_L \cong \hat{\mathcal{S}}_{\mathcal{X}_M, 0}.$$

Our theorem then follows by the Comparison Theorem from the fact that $j^*$ induces isomorphisms on cohomology in degrees 1 and 2 provided $\dim V \geq 4$. To prove this latter fact we consider the following diagram of complexes:

$$
\begin{array}{ccc}
L & \longrightarrow & \Theta_{\mathcal{A}_1^0}(M, E) \\
\uparrow & & \uparrow \\
L_{\text{tan}} & \longrightarrow & \mathcal{L} \\
\end{array}
$$

We have seen that the two vertical arrows are quasi-isomorphisms. But it follows from the discussion in [Y], pp. 81-82, that the top horizontal arrow induces an isomorphism on $H^1$ and $H^2$ in case $\dim V \geq 4$.

The theorem of Buchweitz-Millson and Fujiki-Miyajima can be used to compute $\mathcal{X}_M$ if one has enough information about the ideal of the corresponding singularity. In [M] we use differential geometry to compute $\mathcal{X}_M$ directly for a certain class of $CR$-manifolds. Let $\mathcal{L}$ be a negative line bundle over a compact Kähler manifold $N$ and let $U(\mathcal{L})$ be its unit circle bundle. In [M] we compute $\mathcal{X}_M$ for $M = U(\mathcal{L})$ in case $\mathcal{L}$ is sufficiently negative. Recall that if $\mathcal{L}$ is sufficiently negative, then $\mathcal{L}^{-1}$ gives a projective embedding of $N$ into $\mathbb{C}\mathbb{P}^m$ for some $m$. We then have the following theorem, [M], the $CR$-analogue of Theorem 2 of [Sc2].

**Theorem.** Let $M = U(\mathcal{L})$ as above. If $\mathcal{L}$ is sufficiently negative then $\mathcal{X}_M$ is isomorphic to the parameter space for the versal projective deformation of $N$ in $\mathbb{C}\mathbb{P}^m$. Moreover, by Schlessinger's Theorem, this latter space is isomorphic to the deformation space of the singularity $(\mathbb{C}N, 0)$, where $\mathbb{C}N$ is the affine cone over $N$.

**Example.** Let $N$ be a complex torus of dimension $n$ and $\mathcal{L}$ any negative line bundle over $N$. Then by a direct computation one can show

$$\mathcal{X}_{U(\mathcal{L})} \cong \mathbb{C}^{n(n+1)/2}.$$

In case $\mathcal{L}$ is sufficiently negative, the above Theorem produces an isomorphism between $\mathcal{X}_{U(\mathcal{L})}$ and the parameter space of the versal deformation of $(\mathbb{C}N, 0)$ even though

$$\text{depth}_{(0)} \mathbb{C}N = 2.$$

**Remark.** For any $\mathcal{L}$ as above one can obtain an analytic germ $(\mathbb{C}N, 0)$ by collapsing the zero section. It is reasonable to expect that the parameter space of the versal deformation of $(\mathbb{C}N, 0)$ is again $\mathbb{C}^{n(n+1)/2}$.
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