We will discuss the basic concepts of parametrized Morse theory and show how they are related to the stability theorem, Waldhausen's $A(X)$ and higher Franz-Reidemeister torsion.

§ 1. Standard Morse Theory

In standard Morse theory one usually takes a compact smooth ($C^\infty$) manifold $M$ with boundary the union of three manifolds $\partial_0 M \cup \partial_1 M \cup D \times I$ where $I = [0, 1]$, $D \times 0 = \partial\partial_0 M$ and $D \times 1 = \partial\partial_1 M$. However for the purpose of these notes we will stick mainly to the case when $D$ is empty. We will also write $X = \partial_0 M$. Thus $\partial M = X \bigsqcup \partial_1 M$.

Any generic smooth function $f: M \to I$ with $f^{-1}(i) = \partial_i M$ for $i = 1, 2$ will be a Morse function with a finite number of nondegenerate singularities $x_1, \ldots, x_m$. Each critical point $x_i$ has an index $\text{ind}(x_i) \geq 0$. We can construct a finite relative CW complex $Y = \bigsqcup x_i \bigsqcup e_2 \bigsqcup \cdots \bigsqcup e_m$ with one cell $e_i$ for each critical point $x_i$ and $\dim(e_i) = \text{ind}(x_i)$. From this relative cell complex we can construct the cellular chain complex $C_*(Y, X)$ with coefficients in a ring $R$ or with twisted coefficients given by a locally constant sheaf $\mathcal{F}$ over $M$. If this chain complex is acyclic then we get a torsion invariant $T(Y) = T(M)$ which is an element of some quotient group of $K_1 R$ independent of the choice of the function $f$ (assuming $K_0 Z \to K_0 R$ is a monomorphism).

Example 1.1. $M = S^1 \times I, X = \partial_0 M = \emptyset, \partial_1 M = S^1 \times \{0, 1\}$ and $f$ is the "height" in the following drawing.

In Example 1.1 there are two critical points $x_0, x_1$ of indices 0, 1 and $Y = e_0 \cup e_1$ is a circle. There is no torsion unless we take twisted coefficients in which case $\tau(Y) = \tau(M) = \pm (1 - h^{\pm 1})$ where $h$ is the holonomy of the coefficient sheaf around the circle $S^1 \times 1/2$ in $M$ and the sign ambiguities are due partially to the fact that the 1-cell $e^1$ is unoriented.

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§ 2. Parametrized Morse Theory

The study of parametrized Morse theory began with J. Cerf who used 1 and 2 parameter families of functions on $M = X \times I$ to prove the following theorem with definitions given below.

**Theorem 2.1 [C2].** If $X$ is a simply connected with dimension $\geq 5$ then $\mathcal{C}(X)$ is connected.

**Definition 2.2.** If $X$ is a compact smooth manifold then the concordance space of $X$ is defined to be the space $\mathcal{C}(X) = \text{Diff}(X \times I \text{ rel } X \times 0 \cup \partial X \times I)$ where $\text{Diff}(M \text{ rel } A)$ denotes the space of all self diffeomorphisms of $M$ which are the identity on $A$ with the strong $C^\infty$ topology [C1].

Instead of taking families of functions on a single manifold we will consider functions on “families of manifolds.” This is an equivalent point of view. We take a smooth bundle (i.e. submersion) $p : E \to B$ with fiber $M$ so that $E$ contains the trivial subbundle $X \times B$. If $B = S^{k+1}$, the isomorphism classes of such bundles are in one-to-one correspondence with the elements of $\pi_k \text{Diff}(M \text{ rel } X)$. We are interested in calculating these groups and of finding computable invariants.

Given a smooth bundle $M \to E \to B$ as above we will construct maps:

\[
\theta : B \to |\mathcal{C} . (X)| = \{\text{cell complexes and characteristic maps}\}
\]

\[
\lambda \theta : B \to |\mathcal{K} . (Z, F)| = \{\text{filtered chain complexes}\}
\]

The second map is constructed from the first by composition with a map $\lambda : |\mathcal{C} . (X)| \to |\mathcal{K} . (Z, F)|$. Taking $B = S^{k+1}$ we will get homomorphisms

\[
\theta_* : \pi_k \text{Diff}(M \text{ rel } X) \to \pi_{k+1}|\mathcal{C} . (X)|
\]

\[
\lambda_* \theta_* : \pi_k \text{Diff}(M \text{ rel } X) \to \pi_{k+1}|\mathcal{K} . (Z, F)|.
\]

It turns out that $\theta_*$ is an isomorphism in many cases. Its purpose it to compute the group $\pi_k \text{Diff}(M \text{ rel } X)$. The map $\lambda_* \theta_*$ is used to obtain algebraic invariants to detect particular elements of $\pi_k \text{Diff}(M \text{ rel } X)$. 
The idea behind $\theta$ and $\lambda$ is as follows. We take a "good" function $f : E \to I$ and consider it as a family of functions on the fibers $M_t = p^{-1}(t)$ of $p : E \to B$. This will give us a "family of cell complexes" $Y_t$ which we interpret as a map from $B$ to the "space of all cell complexes." By a "linearization" process we get a map to the "space of all filtered chain complexes." We will explain what these terms mean.

§ 3. Waldhausen's Expansion Category

We use a model for the "space of all cell complexes" due to Waldhausen [W1]. Our nomenclature is however different: what we call the "expansion space" is the loop space of Waldhausen's expansion space. Our expansion space $|\mathcal{E} . (X)|$ is the geometric realization of a simplicial category $\mathcal{E} . (X)$. The objects of $\mathcal{E} \mathcal{O} (X)$, which form the vertices of $|\mathcal{E} . (X)|$, are given by pairs $(Y, \{\psi_e\})$ where $Y$ is a finite relative cell complex $Y = X \cup e_1 \cup \cdots \cup e_m$ and $\psi_e : D^n \to Y$ are characteristic maps (i.e. parametrizations) for the cells. The objects of $\mathcal{E} \mathcal{K} (X)$, which form the $k$-simplices of $|\mathcal{E} . (X)|$, are given by pairs $(Y, \{\psi_e\})$ where $Y$ is a $k$ parameter family of finite relative cell complex $Y = X \times A^k \cup e_1 \times A^k \cup \cdots \cup e_m \times A^k$ and each $\psi_e : D^n \times A^k \to Y$ is a $k$ parameter family of characteristic maps. To avoid set theoretic difficulties we assume that $Y$ is a subspace of a fixed infinite dimensional contractible space, e.g., $\mathbb{R}^\infty$. This also clarifies our assertion that morphism are inclusion maps. Figure 3.1 gives an example for $k = 1$.

![Fig. 3.1](image)

The morphisms of $\mathcal{E} \mathcal{K} (X)$, which may be viewed as gluing maps for the $k$-simplices of $|\mathcal{E} . (X)|$, are given by expansions. Expansions are compositions of elementary expansions which are given by inclusion maps: $Y \to Y \cup D^n \times A^k$ where the $k$ parameter family of disks $D^n \times A^k$ is attached to the $k$ parameter family of cell complexes $Y$ along the southern hemisphere $S^{n-1} \times A^k$. Figure 3.2 shows a path in $|\mathcal{E} . (X)|$ which is obtained by gluing two edges $A, B \in \mathcal{E} \mathcal{K} (X)$ along a morphism $f : \partial_1 A \to \partial_0 B$ from an endpoint of $A$ to an endpoint of $B$. 
An old unpublished result of Waldhausen gives a computation of the homotopy type of some of the components of the expansion space in terms of the Waldhausen $K$-theory of $X$. A proof of this is being prepared for publication in [IW].

**Theorem 3.3 (Waldhausen).** $A(X) \simeq Q(X_+) \times B|\mathcal{E}.h(X)|$ where $A(X) =$ Waldhausen $K$-theory of $X$ [W2], $Q(X_+) = \Omega^\infty \Sigma^\infty (X \coprod_{pt} B|\mathcal{E}.h(X)|)$ is a nonconnected delooping of $|\mathcal{E}.h(X)|$, $\mathcal{E}.h(X)$ is the simplicial full subcategory of $\mathcal{E}$.($X$) consisting of pairs $(Y, \psi)$ where $Y \simeq X$.

§ 4. The Framed Function Theorem

Suppose we have a smooth bundle $M \to E \to B$ as in Section 2 above. Then we would like to construct a map $\theta : B \to |\mathcal{E}$.($X$)|. This is accomplished by the following two theorems.

**Theorem 4.1 [I1].** If $\dim B \leq \dim M$ there exists a smooth function $f : E \to I$ so that for each $t \in B$, the map $f_t : M_t \to I$ given by restricting $f$ to the fiber over $t$ is a generalized Morse function (GMF), i.e. it satisfies the following conditions.

a) $f_t^{-1}(0) = X$, $f_t^{-1}(1) = \partial_1 M_t$,
b) $f_t$ is nonsingular on $\partial M_t$,
c) $f_t$ has only $A_1$ and $A_2$ singularities.

Since $f : E \to I$ is being considered as a family of functions $f_t : M_t \to I$ we consider the fiberwise singular set $\sum(f)$ of $f$ which is defined to be the set of all $x$ in $E$ so that $x$ is a singularity of $f_t : M_t \to I$ where $t = p(x)$ is the image of $x$ in $B$.

**Definition 4.2.** $y \in M$ is an $A_k$ singularity of $g : M \to \mathcal{R}$ if $g$ can be written in local coordinates as

$$g = \pm x_0^{k+1} + \sum_{i=1}^{n} \pm x_i^2 + C.$$
By the Morse lemma $A_1$ singularities are the same as nondegenerate singularities. $A_2$ singularities are also called \textit{birth-death} singularities. This is because an $A_2$ singularity can be perturbed to give either two $A_1$ singularities in “cancelling position” or to give no singularities. This is illustrated in Fig. 4.3. (As before $f$ is the “height” function given by the distance from the bottom of the page.)

Theorem 4.1 is not sufficient for our purposes. Given a smooth bundle $M \to E \to B$ and a smooth function $f: E \to I$ as described in Theorem 4.1 we can do Morse theory on the family of generalized Morse functions $f_t : M_t \to I$ and we get a family of cell complexes parametrized by $B$. However we do not get a family of characteristic maps $\{\psi_e\}$ as required in the definition of the expansion category. Therefore we do not get a map from $B$ into its geometric realization. To remedy this situation we have another theorem.

\textbf{Theorem 4.4} \cite{12} (Framed Function Theorem). In Theorem 4.1 the smooth map $f: E \to I$ can be chosen so that it admits a “framing.” This consists of a Riemannian metric for $E$ and a function $\xi$ on the fiberwise singular set of $f$ of the following form.

\begin{enumerate}
\item[a)] For each critical $x$ of $f_t : M_t \to I$, $\xi(x) = \xi_t(x) = (\xi^1, \ldots, \xi^l)$ is an orthonormal framing for the nonpositive eigenspace of $D^2 f_t(x)$, (i.e. the sum of all eigenspaces of $D^2 f_t(x)$ corresponding to nonpositive eigenvalues).
\item[b)] For each $j$, $\xi^j_t(x)$ is a continuous function on its domain.
\item[c)] At each birth-death singularity the last vector $\xi^l$ lies in $\ker D^2 f_t(x)$ and the intrinsically defined third derivative $D^3 f_t(x)(\xi^l, \xi^l, \xi^l)$ is positive.
\end{enumerate}

Furthermore, if $\dim B < \dim M$ then $(f, \xi)$ is unique up to framed homotopy.

We will call the pair $(f, \xi)$ a \textit{fiberwise framed function} on $E$.

Figure 4.5 illustrates the definition of a framing. In Fig. 4.5.1 there are two $A_1$ singularities of indices 0, 1. Thus a framing associates a unit tangent vector $\xi^1$ to the $A_1$ point of index 1. This goes continuously to the indicated framed $A_2$ singularity. The unit vector $\xi^1$ must point to the right at the $A_2$ singularity by condition (c) in Theorem 4.4. In Fig. 4.5.2 the same $A_1$ singularities with the opposite framing cannot be cancelled in the same way, although there is a more complicated deformation which eliminates these two framed singularities. In Fig. 4.5.3 the framing vector $\xi^2$ plays the role of $\xi^1$ in the first two examples.
Now suppose that $M \to E \to B$ is a smooth bundle and $(f, \zeta): E \to I$ is a fiberwise framed function. Then for each $t$ in $B$ we get a GMF $f_t: E \to I$ and a framing $\zeta_t$. The GMF $f_t$ tells us now to construct a finite relative cell complex $Y_t$ and the framing $\zeta_t$ is exactly what is needed to construct the characteristic maps $\{\psi_t\}$ for $Y_t$. Consequently we get a map $\theta : B \to \partial^+(X)$. The uniqueness of $(f, \zeta)$ up to framed homotopy implies that the map $\theta$ is unique up to homotopy. We note that the dimension condition $\dim B < \dim M$ can be circumvented by "stabilization," i.e. by taking the product with a large disk $D^N$ to get $M \times D^N \to E \times D^N \to B$.

Figure 4.6 gives an example of this construction.
§ 5. Application to Pseudoisotopy

Suppose that $M = X \times 1$. Then smooth bundles $M \to E \to S^{k+1}$ are classified by the homotopy group $\pi_k \mathcal{C}(X)$ and the framed function theorem gives us a homomorphism

$$\theta_* : \pi_k \mathcal{C}(X) \to \pi_{k+1} \mathcal{E}^n(X)$$

This map is an isomorphism if $3k \leq \dim X - 9$ basically because a smooth bundle can be reconstructed from a family of cell complexes by embedding it in Euclidean space and taking a neighborhood. Another way to say this is that we get a highly connected map

$$\theta : B\mathcal{C}(X) \to \mathcal{E}^n \mathcal{h}(X)$$

where $B\mathcal{C}(X)$ is a nonconnected delooping of $\mathcal{C}(X)$. This leads to the following stability theorem. The details can be found in [13].

**Theorem 5.1** (Stability Theorem). The suspension map $\sigma : \mathcal{C}(X) \to \mathcal{C}(X \times I)$ is $\sim \dim X/3$-connected.

Using Theorem 3.1 we also get another proof of the following famous theorem of Waldhausen [W2]:

**Theorem 5.2** (Waldhausen). $A(X) \simeq Q(X_+) \times B^2 \mathcal{P}(X)$ where $\mathcal{P}(X)$ is the direct limit of concordance spaces $\mathcal{C}(X) \to \mathcal{C}(X \times I) \to \cdots \to \mathcal{C}(X \times I^n)$ with respect to suspension and $B^2 \mathcal{P}(X)$ denotes a two-fold nonsimply-connected delooping of $\mathcal{P}(X)$.

The space $\mathcal{P}(X)$ is called the stable pseudoisotopy (or concordance) space of $X$.

§ 6. The Space of Filtered Chain Complexes

This section is a report on recent joint work with John Klein.

One way to obtain $K$-theory invariants for smooth bundles is to triangulate the base space $B$ and associate to each simplex the total singular complex of its inverse image. This gives a functor from the category of simplices of $B$ to a category of chain complexes with additional structure. If this category is carefully constructed it will have homotopy groups closely related to algebraic $K$-groups. One suitable construction is the simplicial category of “filtered chain complexes.”

Suppose that $Z$ is a space, $R$ is an associative ring with 1 and $\mathcal{F}$ is a locally constant sheaf of nonzero finitely generated (f.g.) free $R$-modules over $Z$. Then we construct the “space of filtered chain complexes.” This is the geometric realization of a simplicial category $\mathcal{K}(Z, \mathcal{F})$ whose objects in every degree are quadruples $(S, C, \{S^A\}, \{l_{A,e}\})$ where $S$ is a free $R$-complex, $C$ is a finite poset over $Z$. $\{S^A\}$ is a family of subcomplexes of $S$ indexed by the order ideals in $C$ and $\{l_{A,e}\}$ is a family of cohomology classes indexed by pairs $(A, e)$ where $A$ is an order ideal in $C$ and $e$ is a minimal element of the complement $C \setminus A$ of $A$ in $C$. The precise definition is
rather technical so we will explain it with two examples. We recall that an order ideal in a poset \( C \) is a subset \( A \) of \( C \) so that \( A \) contains any element of \( C \) which is less than any element of \( A \).

**Example 6.1.** Let \( f: M \to I \) be a Morse function as explained in Section 1, let \( \xi \) be a framing for \( f \) and let \( \mathcal{F} \) be any locally constant sheaf of f.g. free \( R \)-modules over \( M \). Then an object \( (S, C, \{ S^A \}, \{ l_{A,e} \}) \) of \( \mathcal{K}_0(M, \mathcal{F}) \) is given as follows.

a) \( S = C_*(M, X; \mathcal{F}) \) is the relative singular complex of \( (M, X) \) with coefficients in \( \mathcal{F} \).

b) \( C = \sum (f) \) is the singular set of \( f \) ordered by critical value. Thus \( a < b \) iff \( f(a) < f(b) \).

i) \( \text{ind}: C \to N \) is given by \( \text{ind}(x) = \text{index of } f \text{ at } x \).

ii) \( p: C \to M \) is the inclusion map.

c) \( S^A = C_*(f^{-1}[0,f(e)] \setminus (C \setminus A), X; \mathcal{F}) \) for any order ideal \( A \) in \( C \) where \( e \) is a minimal element in \( C \setminus A \). (If no such \( e \) exists then \( A = C \) and \( S^C = S \).)

d) If \( A \) is an order ideal in \( C \) and \( e \in A \) is maximal with \( \text{ind}(e) = i \) then let

\[
l_{A,e}: H_i(S^{A \cup e}, S^A) \cong \mathcal{F}_{p(e)}
\]

be the isomorphism specified by the framing \( \xi(e) \) where \( \mathcal{F}_{p(e)} \) is the stalk of \( \mathcal{F} \) over \( p(e) \).

**Example 6.2.** Let \( \pi = \pi_1 X \) and let \( B\pi \) be an open \( K(\pi, 1) \) neighborhood of \( X \) in \( R^\infty \). Let \( \mathcal{F} \) be any locally constant sheaf of f.g. free \( R \)-modules over \( B\pi \). Let \( \mathcal{E}.\, \pi(X) \) denote the simplicial full subcategory of \( \mathcal{E}. \, (X) \) whose objects are pairs \( (Y, \psi) \) so that \( Y \) is a subspace of \( B\pi \). Then there is a simplicial functor \( \lambda: \mathcal{E}.\, \pi(X) \to \mathcal{K}. \, (B\pi, \mathcal{F}) \) given in degree zero by \( \lambda(Y, \psi) = (S, C, \{ S^A \}, \{ l_{A,e} \}) \) where \( S, C, \{ S^A \}, \{ l_{A,e} \} \) are given as follows.

a) \( S = C_*(Y, X; \mathcal{F}_Y) \) where \( \mathcal{F}_Y \) is the restriction of \( \mathcal{F} \) to \( Y \).

b) \( C \) is the set of cells of \( Y \). (The indexed poset \( C \) is actually part of the structure of \( (Y, \psi) \).)

i) \( \text{ind}: C \to N \) is given by \( \text{ind}(x) = \text{dim}(x) \).

ii) \( p: C \to B\pi \) is given by \( p(x) = \psi_x(0) \).

c) \( S^A = C_*(Y^A, X; \mathcal{F}_{Y^A}) \) where \( Y^A \) is the closed subcomplex of \( Y \) corresponding to \( A \).

d) If \( A \) is an order ideal in \( C \) and \( e \in A \) is maximal with \( \text{ind}(e) = i \) then let

\[
l_{A,e}: H_i(S^{A \cup e}, S^A) \cong \mathcal{F}_{p(e)}
\]

be the isomorphism specified by the characteristic map \( \psi_e \).

Let \( \mathcal{K}.\, h(Z, \mathcal{F}) \) denote the simplicial full subcategory of \( \mathcal{K}. \, (Z, \mathcal{F}) \) whose objects are those quadruples \( (S, C, \{ S^A \}, \{ l_{A,e} \}) \) so that \( S \) is acyclic. Assume that \( Z \) is connected and that the natural map \( K_0Z \to K_0 R \) is a monomorphism. Then using ideas from the proof of Theorem 3.3 we obtain the following.

**Theorem 6.3** [IK]. *There is a homotopy fiber sequence:*

\[
|\mathcal{K}.\, h(Z, \mathcal{F})| \to Q(Z_+) \to \mathbb{Z} \times B \text{Gl}(R)^+.
\]
Corollary 6.4 (Higher Franz-Reidemeister Torsion). Suppose that $\pi_1 M = \pi_1 X = \pi$ is a finite group, $R = \mathbb{C}$ and $\mathcal{F}$ has unitary holonomy, i.e. $\mathcal{F}$ is given by a unitary representation of $\pi$. Let $M \to E \to S^{2k}$ be a smooth bundle as in Section 2 so that the map in homology $H_*(M, X; \mathcal{F}) \to H_*(E, X \times S^{2k}; \mathcal{F})$ induced by the inclusion of $M$ in $E$ is a monomorphism. Then there is a naturally defined torsion invariant $\tau(E) \in \mathbb{R}$. Furthermore these invariants detect the rational homotopy groups of $\mathcal{P}(*)$ and are thus nontrivial for all even $k$.

Remark. Higher Franz-Reidemeister torsion was first constructed by Wagoner [W] but only for bundles over circles. Klein [K] was the first to construct higher Franz-Reidemeister torsion invariants for bundles over higher dimensional spheres but he assumed that $H_*(M, X; \mathcal{F}) = 0$.

References

[IW] Igusa, K., Waldhausen, F.: The expansion space and its application to pseudoisotopy theory. In preparation