Localization and Intermittency: New Results

Stanislav A. Molchanov

Department of Mathematics and Mechanics, Moscow State University, Moscow, 119808, USSR

The theory of the Anderson localization was developed for a long time in the initial probabilistic framework for the operators of the form

$$H = \Delta + V(x, w), \quad x \in \mathbb{R}^d (\mathbb{Z}^d), \quad w \in (\Omega, F, P).$$

Here $\Delta$ is the Laplacian (continuous or on lattice) and $V(x, w)$ is the random homogeneous field (or process) on a probability space $(\Omega, F, P)$. The central achievement of one-dimensional theory was a series of S. Kotani's articles [1]–[3], where he discovered deep connections between Lyapunov's exponents $\gamma(\lambda)$, the structure of the prediction of the potential $V(\cdot)$ and the spectral localization of $H$.

Techniques of the cluster resolvent expansions, developed by Fröhlich and Spencer [4], together with generalization of Kotani's idea [2], proposed almost simultaneously (in different forms by Souillard et al. [5] and Simon-Wolff [6]) made it possible to solve the point spectrum problem for Anderson's tight-binding model (1) in the multidimensional lattice case $\mathbb{Z}^d, d > 1$.

These results are summed up in the review article of Martinelli-Scoppola [7]. Simplifications and generalizations were proposed recently in this field by Dreifus and Klein [8].

Probabilistic approach, nevertheless, did not clear up the central physical idea of localization—the absence of resonance between a quantum particle with a given admissible energy $\lambda$ and some (rich enough) family of the blocks of the potential. It's natural to attempt to formulate the direct geometric conditions for the individual potential $V(\cdot)$, which will lead to the localization of the spectrum. The first and a very important step in this direction was made in the paper Simon-Spencer [9]. They proved (in the one-dimensional case) that unboundness of potential $V$ gives us the singular spectrum.

Moreover, in a multidimensional situation ($d > 1$), even in the lattice case, there are many open problems on the thin structure of the spectrum $H$, and on the existence of bifurcations with respect to some parameters of the model, etc. It's related, for example, to the simplest functional of $\sigma(H)$, namely the integral density of states $N(\lambda)$ (the so-called problem of Lifshitz tails).

In this lecture I'll give an account of some recent results in this direction. It is based on the papers, which I wrote together with my friends and colleagues, especially J. Görnter, A. Gordon, L. Pastur and B. Simon.

This lecture was prepared at the time of my visit to Caltech (spring 1990) and I am very grateful to Barry Simon for his hospitality, to A. Klein, R. Carmona,
§ 1. One-Dimensional Localization. Lattice Case

The next group of results can be obtained by the combination of the ideas of Simon-Spencer [9] (appearance of the family of non-resonant blocks), Fröhlich-Spencer [4, 7] (cluster expansions of resolvent) and some new considerations, the principal of which is a randomization of energy.

**Theorem 1.** Consider in $l^2(\mathbb{Z}_+)$, $\mathbb{Z}_+ = \{0, 1, \ldots\}$ the Schrödinger operator $H^\theta = \Delta + V(x)$, $x \in \mathbb{Z}_+$. $A$ is a discrete Laplacian, $A\psi(x) = \psi(x+1) + \psi(x-1)$ and the parameter $\theta$ is a boundary phase connected with the boundary condition $\psi(-1) \cos \theta + \psi(0) \sin \theta = 0$. Assume, that for some energy interval $I \subset \mathbb{R}$ there exists the sequence of the "nonresonant blocks" $[x_n, y_n]$, $n = 1, 2, \ldots$, that is a sequence of points $0 < x_1 \leq y_1 < x_2 \leq y_2 \ldots$ such that for every $\lambda \in I$

$$|R_\lambda^{(n)}(x_n, y_n)| \leq \delta_n; \quad \delta_n \to 0, \quad n \to \infty. \quad (2)$$

Here $R_\lambda^{(n)} = (H^\theta - \lambda)^{-1}$ is the resolvent of the operators on the block $[x_n, y_n]$: $H^\theta_{\psi} = A\psi + V(x)\psi(x)$, $x_n \leq x \leq y_n$, $\psi(x_n - 1) = \psi(y_n + 1) = 0$.

Suppose that there exists a nondecreasing sequence of constants $A_n \geq 1$ and constant $c \geq 1$ such that

$$\sum_n A_n \delta_n < \infty, \quad L_n = |y_{n+1} - x_n| + 1 \leq A_1 \ldots A_n \cdot c^n. \quad (3)$$

Then $\sigma(H^\theta) \cap I = \sigma_{pp}(H^\theta) \cap I$ almost everywhere (a.e.) in $\theta$.

In the particular case, which is important for many applications, when $x_n \leq c^n$ (coordinates of blocks do not increase faster than exponential) conditions (2) and (3) become simple. It is enough to require $\sum_n \delta_n < \infty$.

Effective verification of (2) and (3) gives the next result. It is based on the physical idea of the absence of resonance.

**Theorem 2.** Suppose (under the conditions of Theorem 1) that for every $\lambda \in \mathbb{R}$ there exist a positive constant $\rho = \rho(\lambda)$ and a sequence of non-resonant blocks $[x_n, y_n](\lambda)$, such that $\text{dist}\{\lambda, \sigma(H^{(n)}(\lambda))\} \geq \rho(\lambda)$ and $l_n = |x_n - y_n| + 1 \geq A(\rho) \ln n$ where the function $A = A(\rho)$ could be specified. Then, if $x_n < c^n$, $c > 0$,

$$\sigma(H^\theta) = \sigma_{pp}(H^\theta) \quad \text{a.e. in } \theta. \quad$$

Theorems 1 and 2 include all well-known mechanisms of localization: high barriers, long bumps, gaps in periodic potentials, etc. (See for comparison [9].)

The typical examples where Theorems 1 and 2 could (and will) be applied are random (usually nonhomogeneous) potentials $V(x, w)$, $x \in \mathbb{Z}_+$.

**Example 1** (Independent Random Variables). Let $V(x) = \xi_x(w)$, $x \geq 0, w \in (\Omega, F, P)$ be i.r.v and $\xi_x(w) = a(x) + \eta_x(w), x > 0$, where $a(x)$ is an arbitrary
nonrandom function and \( \{\eta_x(w), x \geq 0\} \) is uniformly non-degenerate in the following sense: There exist positive constants \( \varepsilon_0, \delta_0 \) such that

\[
P\{\eta_x > 1 + \delta_0\} \geq \varepsilon_0, \quad P\{\eta_x < -1 - \delta_0\} \geq \varepsilon_0, \quad x = 0, 1, \ldots
\]

Then \( \sigma(H^0 = A + \xi_x) = \sigma_{pp}(H^0) \) a.s. \( P \) and a.e. in \( \theta \).

**Example 2** (Gaussian Potentials). Let \( V(x) = \xi_x, x \geq 0 \), be a nonstationary gaussian sequence, \( \langle \xi_x \rangle = 0, 0 < c_1 < \langle \xi_x^2 \rangle < c_2 < \infty, \) \( \text{cov}(\xi_x, \xi_y) = \langle \xi_x \xi_y \rangle \leq \frac{\alpha}{\ln^{1+\varepsilon}|x-y|} \), \( |x-y| \geq c_4 \). The last condition is well known in the theory of gaussian fields and processes: if correlations decay only logarithmically, then the structure of the high peaks suffers from bifurcations. It’s possible to prove that in our case, \( P \)-a.s. in any interval \( [2^n, 2^{n+1}) \), \( n > n_0(w) \), there exists a “triplet” \( |\xi_x| > \delta \sqrt{n}, |\xi_{x+1}| > \delta \sqrt{n}, |\xi_{x+2}| > \delta \sqrt{n} \) of the high peaks. It’s enough to allow us to apply Theorem 2 and so \( \sigma(H^0) = \sigma_{pp}(H^0) \) a.e. in \( \theta \) and \( P \)-a.s.

**Example 3** (Unbounded Quasiperiodic Potentials). Let \( f(t) = f(t+1) \) be periodic function on the unit circle \( S^1 \) for which there exists at least one point \( t_0 \) of logarithmic singularity:

\[
|f(t)| \geq c \left( \ln |t-t_0| \right)^{1+\varepsilon}, \quad t \in S^1, \quad \varepsilon > 0.
\]

Then for quasirandom potentials of the form

\[
V(x) = f(ax), \quad V(x) = f(ax^2 + \beta)
\]

for almost all values of parameters \( a, (a, \beta) \) we do have a pure point spectrum (a.e. in \( \theta \)). The second of these potentials (for the case \( f(t) = \text{ctg} \pi t \)) is the popular model of “quantum chaos” (see [18]).

For the case of entire axis \( \mathbb{Z}^1 \), the treatment for the operator

\[
H = A + V(x), \quad -\infty < x < \infty
\]

is more subtle. Of course, if conditions of Theorems 1 and 2 are satisfied for \( x > 0 \) and \( x < 0 \), then it’s possible to prove (as in Theorems 1 and 2) that for a.e. \( \lambda \in I \) (or \( \lambda \in R^1 \))

\[
R_{\lambda}(0,x) = (H - \lambda)^{-1}(0,x) \in l^2(\mathbb{Z}^1).
\]

In this spectral problem there is no exterior random parameter (such as \( \theta \) in the case \( \mathbb{Z}^1 \)) and (following [5] and [6]) the randomness must be included in the potential. The next lemma generalizes the one-dimensional results of [5] and [6].

**Lemma 1.** Let \( H^\varphi = A + V_0(x) + a\varphi(x), \) \( x \in \mathbb{Z}^1 \), for a.e. \( \lambda < R^1 \) and denote by

\[
R_{\lambda}^\varphi(0,x) = (H + V_0 - \lambda)^{-1}(0,x) = (H^\varphi - \lambda)^{-1}(0,x) \in l^2(\mathbb{Z}^1).
\]

If perturbation \( \varphi \) decays “sufficiently fast”, then for a.e. \( \lambda \) any two Weyl’s solutions \( \psi_{\lambda}^\pm(x) \) of the equation \( (H^\varphi - \lambda)\psi = 0 \) satisfy

\[
\sum_{x \in \mathbb{Z}^1} |\psi^-||\psi^+(x)||\varphi(x)| < \infty
\]
Then, a.e. in a, \( \sigma(H^a) = \sigma_{pp}(H^a) \). The typical \( \psi \) is the one which satisfies \( |\varphi(x)| \leq \exp(-\varepsilon|x|) \).

In [5] and [6] similar results were proved for functions \( \varphi \) with finite support.

We will give two examples of the above lemma.

**Example 4.** Let \( V(x) = \xi_x, \ x \in \mathbb{Z}^1 \) be i.r.v. which satisfy the conditions of Example 1. If, in addition, one of them, say \( \xi_0 \), has absolutely continuous distribution, then

\[
\sigma(\Delta + \xi_x) = \sigma_{pp}(\Delta + \xi_x) \quad P-a.s.
\]

**Example 5.** Let \( \xi_x, \ x \in \mathbb{Z}^1 \) be i.i.d.r.v. with common absolutely continuous distribution \( \langle \xi_x \rangle < \infty \) and \( \varphi(x) \), such that \( |\varphi(x)| \leq \exp(-\varepsilon|x|) \), be the elementary potential. Consider a homogeneous random potential of alloy type

\[
V(x) = \sum_{n=-\infty}^{+\infty} \xi_n \varphi(x-n).
\]

It’s not very difficult to verify the application of Theorem 2 in this case (see [17], where similar problems were analyzed in a more complicated situation). Application of Lemma 1 to the “partition”

\[
V(x) = \sum_{n \neq 0} \xi_n \varphi(x-n) + \xi_0 \varphi(x) = V_0(x) + \xi_0 \varphi(x)
\]

shows that the Schrödinger operator \( H = \Delta + \sum_n \xi_n \varphi(x-n) \) under the above conditions has \( P \)-a.s. p.p. spectrum in \( l^2(\mathbb{Z}^1) \). Earlier (see [17]) it was known only that \( \sigma(H) = \sigma_{\text{sing}}(H) \) \( P \)-a.s.

### § 2. Some Generalizations

The main idea of Theorems 1 and 2 can be extended to more general one-dimensional lattice systems: Jacobi operators of the form \( H \varphi(x) = l(x-1)\varphi(x-1) + l(x)\varphi(x+1) \), operators where the Laplacian \( \Delta \) is replaced by nonlocal convolution \( \sum_y l(x-y)\varphi(y) = A\varphi(x) \) and kernel \( l(x-y) \) decays fast enough off the diagonal and so on.

Let us formulate two theorems of this type. The first result about the “random string” has a clear mechanical meaning.

**Theorem 3.** Let \( H^\theta \varphi(x) = l(x-1)\varphi(x-1) + l(x)\varphi(x+1), \ l(x) > 0, \ x \geq 0 \) be the operator of the lattice string with boundary condition \( \varphi(-1) \cos \theta + \varphi(0) \sin \theta = 0, \ \theta \in (0, \pi) \). Assume that for some sequence of blocks \( [x_n, y_n] : \ l_n = |x_n - y_n| + 1 > c \ln^{1+s} n, c > 0 ; \ x_n < c_1, \ c_1 > 1 \) we have \( l(x) \leq \lambda_0, \ x \in [x_n, y_n], \ n = 1, 2, \ldots \). Then a.e. \( \theta \in [0, \pi) \)

\[
\sigma(H^\theta) \cap (|\lambda| > \lambda_0) = \sigma_{pp}(H^\theta) \cap (|\lambda| > \lambda_0).
\]
Physically this means that "long, not very elastic inclusions in an elastic medium" lead to localization of short waves. In some cases it is possible to prove that for $| \lambda | < | \lambda_0 |$ there is no p.p. spectrum.

The following is connected with nonlocal Laplacian.

**Theorem 4.** Let $H = \tilde{A} + V(x)$, where $\tilde{A} \varphi(x) = \sum_{y \in \mathbb{Z}} I(x - y) \varphi(y)$, $|I(z)| \leq \frac{c}{1 + |z|^{\beta}}$, $\beta > 8$ and $V(x)$ is i.i.d.z.v. with common a.e. distribution. Then $P$-a.s.

$$\sigma(H) = \sigma_{pp}(H).$$

Other generalizations referred to increasing the dimension.

**Schrödinger Operator in the Strip.** Let $D = \mathbb{Z}_{+} + \mathbb{Z}_N$, where $\mathbb{Z}_N = (0, 1, \ldots, N - 1)$, $N \equiv 0$, is a group of residue mod $N$ and hamiltonian $H$ in $l^2(D)$ has a form

$$H^0 = A + V(x, z), \quad (x, z) \in D, \quad A \varphi(x, z) = \varphi(x + 1, z) + \varphi(x - 1, z) + \varphi(x, z + 1) + \varphi(x, z - 1), \quad \cos \theta_x \varphi(-1, z) + \sin \theta_x \varphi(0, z) = 0, \quad z \in \mathbb{Z}_N,
\theta = (\theta_0, \ldots, \theta_{N-1}) \in [0, \pi)^N. \quad (6)$$

It is well known, that for homogeneous fields $V(x, z)$ (with group of shifts $x \rightarrow x + h$, $x, h \in \mathbb{Z}_1$) the localization theorems may be proven by the classical method of Ljapunov exponents. This method in the case of the strip is more complicated than for $\mathbb{Z}_1$. On the contrary, the cluster method of Section 1 doesn't feel the difference between $\mathbb{Z}_1$ and $\mathbb{Z}_1 \times \mathbb{Z}_N$.

**Theorem 5.** Assume that for given operator $H^0$ and energy interval $I \subset \mathbb{R}_1$ there exists a system of blocks $B_n = \{(x, z) : (x - n \leq x \leq y_n, z \in \mathbb{Z}_N), \quad n = 1, 2, \ldots$ and constants $h_n, \varrho_n$, such that

- $a) \quad V(x, z) \geq h_n, \quad (x, z) \in B_n$
- $b) \quad \text{dist}(h_n, I) = 4 + \varrho_n, \quad \varrho_n > 0.$

If $x_n < c^n$, $c > 1$, $l_n = |x_n - y_n| + 1$ and $\sum_n (1 + \varrho_n)^{-l_n} < \infty$, then

$$\sigma(H^0) \cap I = \sigma_{pp}(H^0) \cap I \quad \text{a.e. } \theta \in [0, \pi)^N.$$

Note that the central conditions

$$\text{dist}(h_n, I) \geq 4 + \varrho_n, \quad \varrho_n > 0, \quad n = 1, 2, \ldots$$

have a slightly different form with respect to "pure one-dimensional" theory. Early on we used $\text{dist}(h_n, I) > 2 + \varrho_n$. This is because $\sigma(A) = [-2, +2]$ in $l^2(\mathbb{Z}_1)$ and $\sigma(A) = [-4, 4]$ in $l^2(\mathbb{Z}_1 \times \mathbb{Z}_N)$.

In virtue of Theorem 5 and its generalization to the case of $(\mathbb{Z}_1 \times \mathbb{Z}_N)$, which uses simple variants of Lemma 1, all previous examples (1-5) automatically transfer to the corresponding examples in the strip (half-strip). (The number of a.e. conditions now equals $N$. For example, in the analog of Lemma 1 we must
consider $N$ perturbations: $\xi_1\varphi_1(x, z_1), \xi_2\varphi_2(x, z_2), \ldots, \xi_N\varphi_N(x, z_N); (\xi_1, \ldots, \xi_N)$ has a.e. distribution.)

Multidimensional generalizations $(d > 1)$ of Theorems 1 and 2 of Section 1 exist, but are noneffective in the case of homogeneous potentials. They act as follows: For any $n = 1, 2, \ldots$ in $\mathbb{Z}^d$ let there exist two 1-connected (in the sense of percolation theory) surfaces $\Gamma_n^+, \Gamma_n^-, D_n = \text{diam} \Gamma_n^-, d_n = \min |x - y|_{x \in \Gamma_n^+, y \in \Gamma_n^-}$, $\text{Int} \Gamma_1^+ \subset \text{Int} \Gamma_2^+ \subset \text{Int} \Gamma_2^+ \subset \ldots$. Let $|\Gamma_n^+| = O(D_n^{-1}), |\text{Int} \Gamma_n^+| = \frac{1}{2}(D_n^d), n \to \infty$ and for potential $V(x)$, $x \in \mathbb{Z}^d$ the next estimation takes place:

$$|V(x)| \geq c(D_n^{-1}n^{1+\epsilon})^{1/d_n}, \quad x \in B_n = \text{Int} \Gamma_n^+ \setminus \text{Int} \Gamma_n^-.$$

Under these conditions the cluster expansion of resolvent $R^0_\lambda(0, x) = (\lambda + V(x))^{-1}(0, x)$, with respect to non-resonant families of blocks $B_n(\partial B_n = \Gamma_n^+, \Gamma_n^-)$, shows that

$$R_\lambda(0, x) \in l^2(\mathbb{Z}^d) \quad \text{a.e.} \, \lambda.$$

Using the Simon-Wolff theorem under some additional technical conditions it is possible to prove a few concrete results about the p.p. spectrum.

**Example 6.** Let $V(x) = \xi_x|x|^\alpha, \quad x \in \mathbb{Z}^d, \alpha > 0$ and $\xi_x$ is i.i.d.r.v. with some moments properties and a.e. distribution. For example, $\xi_x \in [-1, 1]$ or $[0, 1]$ and is uniformly distributed. Another case is where $\xi_x$ is standard $N(0, 1)$ gaussian r.v. Here $P$-a.s. for all $\alpha > 0$

$$\sigma(\lambda + V(x)) \leq \sigma_{pp}(\lambda + V(x)).$$

The corresponding eigenfunctions decay superexponentially. The structure of the $\sigma_{\text{ess}}$ in the case $\xi_x \in [0, 1]$ is not trivial and dependent on $\alpha$. Note that $\sigma(\lambda + V(x)) = \sigma_{\text{disc}}(\lambda + V(x))$ if $\alpha > d$.

**Example 7.** There exist potentials $V(x, w), \quad x \in \mathbb{Z}^d, \quad d > 1$ such that

a) $V(x, w)$ is homogeneous and ergodic with “good” mixing properties. For example it has strong mixing condition with respect to the family of bounded subsets $\mathbb{Z}^d$.

b) $\langle |V(x, w)|^p \rangle < \infty$ for every $p > 0$.

c) Operator $H = \lambda + \sigma V(x)$ has p.p. spectrum for all positive coupling constants $\sigma$.

Note, however, that in this “counterexample” to the Anderson’s hypothesis the correlations of $V(x, w)$ decay (in any sense) very slowly. Potentials $V(x, w)$ of this example do not percolate from above. That is, for any $\lambda > 0$ the set $A^- (\lambda) = \{x : |V(x, w)| < \lambda\}$ is the union of finite components. Typical potentials (in physical applications) have a finite level of percolation [16].
§ 3. Continuous One-Dimensional Case

In the transition from the results of Section 1 for lattice Laplacians to their analogs for the operator

\[ H^0 \psi(x) = -\frac{d^2 \psi}{dx^2} + V(x), \quad x \geq 0; \quad \psi(0) \cos \theta + \psi'(0) \sin \theta = 0 \]

there are a few technical obstacles connected with unboundness of $-\Delta = -d^2 / dx^2$ in $l^2(R^1_+)$. If $V(x) \geq 0$ the methods and results are similar to Theorems 1–3 but seem stronger.

**Theorem 6.** Let potential $V(x) \geq 0$ for some $\delta_0 > 0$ have a next estimation from below. For some sequence $x_n \uparrow \infty$, $x_{n+1} - x_n \uparrow \infty$

\[ V(x) > h_n, \quad x \in [x_n, x_n + \delta_0]. \quad (8) \]

If

\[ \lim_{n \to \infty} \frac{x_{n+1}}{x_n} \exp(-\delta_0 \sqrt{h_n}) < 1, \quad \sum_n \exp(-\delta_0 \sqrt{h_n}) < \infty \quad (8') \]

then

\[ \sigma(H^0) = \sigma_{pp}(H^0) \quad \text{a.e. } \theta \in [0, \pi). \]

A. Gordon has analyzed in detail the particular case $V(x) \equiv 0$, $x \notin \cup_n [x_n, x_n + \delta_0]$, $V(x) = h_0 \uparrow \infty$, $x \in [x_n, x_n + \delta_n]$ (rear high scatterers). He proved that in this case under condition

\[ \lim_{n \to \infty} \frac{x_{n+1}}{h_n \cdot x_n} \exp(-\delta_0 \sqrt{h_n}) > 1 \quad (9) \]

\[ \text{Sp} H^0 = \text{Sp}_{sc} H^0 \quad \text{for a.e. in } \theta. \]

The next result is a variant of Kotani's theorem [3], but for p.p. spectrum (but not singular as in [3]).

**Theorem 7.** Let $F_0(x) = F_0(x + T)$ be a continuous periodic function and $[x_n, y_n]$ be a sequence of blocks such that $\sup_{x \in [x_n, y_n]} |V(x) - F_0(x)| \to 0$ as $n \to \infty$, $x_n < c^n$, $c > 1$, $l_n = |x_n - y_n| \geq \ln^{1+\epsilon} n$, $\epsilon > 0$, $n \geq n_0$. If $\Delta$ is one of the gaps in the $\sigma_{ess}(-d^2 / dx^2 + F_0(x))$, then for a.e. in $\theta$

\[ \sigma(H^0) \bigcap \Delta = \sigma_{pp}(H^0) \bigcap \Delta. \]

If potential $V(x)$ is unbounded from below, but has a logarithmic estimation of the form $V(x) \geq -c_1 \ln(|x| + 1) + c_2$, then it is possible to prove variants of Theorems 5 and 6 under stronger assumptions on the increase of $\{h_n\}$ in Theorem 5 or $\{l_n\}$ in Theorem 6.

In the case of the entire line $R^1$ (that is in $l^2(R^1)$) the continuous analog of Lemma 1 (Sect. 1) can be used. It makes it possible to prove the theorems on the point spectrum for many physically interesting models.
Example 8. Let \( V(x) = \sum_{n=-\infty}^{+\infty} \xi_n \varphi \left( \frac{x-x_n}{b_n} \right) \) be a "shot noise" potential. Here \( \{x_n\} \) is the Poissonian points flow, \( |\varphi(x)| \leq \exp(-c|x|) \) the elementary potential, \( \{\theta_n, \xi_n\} \) i.i.d. r.v. vectors with finite exponential moments: \( \exp(z\xi) < \infty, \exp(\theta z) < \infty, \exp(z^{1/2} < \infty, |z| < z_0, z_0 > 0 \) and the distribution of \( \xi_n \) is a.c. Then P-a.s. in \( l^2(\mathbb{R}^1) \)

\[
\sigma(H) = \sigma \left( -\frac{d^2}{dx^2} + V(x, w) \right) = \sigma_{pp}(H).
\]

This example is closely connected with the paper [17] in which the authors proved that in the same situation \( \sigma(H) = \sigma_{\text{sing}}(H) \) P-a.s., but under additional restrictions, the elementary potential \( \varphi \) is not the soliton. Although this condition is not essential and the spectrum is p.p., the appearance of the soliton in this context is not accidental.

Theorem 8. There exists potential \( V(x, w), x \in \mathbb{R}^1 \) and large parameter \( L > 0 \) for which

a) \[
\left| V(x, w) - \sum_{n=-\infty}^{+\infty} \left( 2 \xi_n^2 \right) \right| \leq e^{-\delta L}, \quad \delta > 0, \quad x \in \mathbb{R}^1.
\]

Here \( \{\xi_n\} \) is i.i.d. r.v. (for instance uniformly distributed in \([0, a], a > 0\), \( \{l_n\} \) is a homogeneous random point process in \( \mathbb{R}^1 \) (dependent on \( \{\xi_n\} \)) with good mixing properties. Note that \( \varphi(x) = -2/ch^2 x \) is the simplest soliton (1-soliton).

b) \[
\sigma(H) \bigcap [0, \infty) = \sigma_{ac}(H) \bigcap [0, \infty), \quad \sigma(H) \bigcap [-\infty, 0] = \sigma_{pp}(H) \bigcap [-\infty, 0].
\]

This potential is one of the realizations of "soliton's gas". It is closely connected with the problem of statistical solutions of the KdV-equation.

§ 4. Parabolic Problems for the Anderson Model. Intermittency and Related Topics

Evolution problems for the physical fields in the random medium (chemistry kinetics, hydrodynamics, etc.) very often have the form of a parabolic equation with random coefficients, in particular, with random potential. The simplest example is

\[
\frac{\partial c}{\partial t} = D \Delta c + \xi(x)c
\]

\[
c(0, x) \equiv 1.
\]

Function \( c(t, x), t \geq 0 \) has the meaning of the concentration of the particles at moment \( t \) in the point \( x \). The kinematic part of the hamiltonian \( D \Delta \) describes its diffusion (\( D \)-diffusion coefficient) and potential \( \xi(x) \) its transformation. If \( \xi(x) > 0 \), then \( \xi(x)dt \) is the probability that in the time interval \((t, t + dt)\) any particle in the point \( x \) will split (birth of a particle). If \( \xi(x) < 0 \), then \( \xi(x) \) is the intensity of the death process in the point \( x \). We will consider problem (11) in the discrete case where \( x \in \mathbb{Z}^d, D \Delta \varphi(x) = D \sum_{|x'-x|=1} (\varphi(x') - \varphi(x)) \) is the lattice.
Laplacian (which is the generator of the symmetrical random walk $x_t$, $t \geq 0$ in continuous time) and $\xi(x, w)$ is the homogeneous ergodic random field.

If $V(x) = \xi_x$, $x \in \mathbb{Z}^d$ are i.i.d.r.v., then $H - D\Delta + \xi$ is the hamiltonian of the tight-binding Anderson model. The diffusion coefficient $D$ is the inverse coupling constant $D = 1/\sigma$. It's well known [7], [8], that for $\sigma \gg 1$ (strong disorder) or for small $\sigma$, but $\lambda \gg 1$ (fluctuation part of the spectrum) under some technical restrictions $P$-a.s. $\sigma(H) = \sigma_{pp}(H)$.

**Asymptotic Properties.** The solution $c = c(t, x)$, $t \to \infty$ can be represented in the spectral terms. It allows investigation by direct probabilistic methods. This gives us essential information about the structure $\sigma(H)$, $\lambda \gg 1$.

The central qualitative property of the field $c(t, x)$, $t \to \infty$ is its intermittency – that is, informally, the existence of strongly pronounced spatial structures (in this case sharp and high peaks). The definition of intermittency is given in terms of statistical moments $c(t, x)$.

Remember that $c(0, x) = 1$. This means that $c(t, x)$ is a homogeneous ergodic field for every $t > 0$. It is not very difficult to prove that the condition $(\exp(t\xi)) < \infty$, $t > 0$ is necessary and sufficient for the existence of $\langle c^p(t, x) \rangle$, $t > 0$, $p = 1, 2, \ldots$.

**Definition 1.** We will say that the family of the fields $c(t, x)$, $x \in \mathbb{Z}^d$, $t > 0$ is the asymptotic intermittency parameter of the family when $t \to \infty$, if for the functions

$$\wedge_p(t) = \ln \langle c^p(t, \cdot) \rangle$$

the following relations take place as $t \to \infty$

$$\wedge_2(t) \ll \frac{\wedge_2(t)}{2} \ll \frac{\wedge_B(t)}{B(t)} \ll \ldots$$

Here $A(t) \ll B(t)$ means $B(t) - A(t) \to_0 + \infty$.

**Theorem 9.** If potential $\xi(x)$ is unbounded from above and $G(t) = \langle \exp(t\xi(\cdot)) \rangle < \infty$, $t > 0$, then the solution $c(t, x)$ is asymptotically intermitted in the sense of Definition 1. The logarithmic asymptotics of the statistical moment has a form

$$\frac{\ln \langle c^p(t, 0) \rangle}{p} = \frac{\wedge_p(t)}{p} \sim_{t \to \infty} \frac{G(pt)}{p}.$$  \hspace{1cm} (12)

The more exact asymptotics depends upon the structure of the tails of the one-dimensional distributions of potential $\xi(\cdot)$.

**Theorem 10.** Let $\xi(x)$, $x \in \mathbb{Z}^d$ be i.i.d.r.v. (Anderson model) and $P\{\xi(1) > t\} \sim_{t \to \infty} \exp(-ct^\beta)$, $\beta > 1$. Then

$$\frac{\wedge_p(t)}{p} = \frac{G(pt)}{p} - 2dDpt + \bar{\omega}(t) = \frac{c(\beta)t^{\beta-1}}{p} - 2dDpt + \bar{\omega}(t), \quad t \to \infty. \hspace{1cm} (13)$$

But if $P\{\xi(\cdot) > t\} \sim \exp(-c\exp(c, t^\beta))$, $\beta > 1$, then

$$\frac{\wedge_p(t)}{p} = \frac{G(pt)}{p} + \bar{\omega}(t) = c(\beta, p) \ln \frac{t}{\bar{\omega}} + \bar{\omega}(t). \hspace{1cm} (14)$$
The difference between these two formulas are due to physical reasons. In the first case “strong centers” of the potential $\xi(\cdot)$, which contribute mainly to the growth of the number of particles, have the form of a single high peak. In the second case it is wide but not very high islands. The second term of asymptotics of $\wedge_t(t)/p$ describes the probability of “keeping” particles by “strong centers”.

It is possible to observe the same effect in the results about the almost sures (a.s.) behavior of $c(t, x)$, $t \to \infty$.

**Theorem 11.** Let $\ln \ln P\{\xi > t\}^{-1} < c_1 t^\beta$, $\beta < 1, c_1 > 0$ (roughly speaking, $P\{\xi > t\} \geq \exp\{-c \exp(c_1 t^\beta)\}, \beta < 1$. Then $P$-a.s. $t \to \infty$

$$\frac{\ln c(t, x)}{t} = \left(\log\left(\frac{1}{H(0)}\right)\right)^{-1} (d \ln t) - 2dD + O(1). \quad (15)$$

(Here, $H(t) = P\{\xi > t\}$, $(\cdot)^{-1}$ means inverse function.)

If, however,

$$\ln \ln H(t) > c_2 t^\beta, \quad \beta > 1, \quad c_2 > 0 \quad (16)$$

then $P$-a.s.

$$\frac{\ln c(t, x)}{t} = \left(\log\frac{1}{H(\cdot)}\right)^{-1} (d \ln t). \quad (17)$$

Consider now the initial parabolic Anderson problem (11) for the localized initial condition $c(0, x) = \delta_0(x)$. Assume, that $\xi(x) > 0$ is an i.i.d.r.v. with “exponential tails”. For simplicity let $P\{\xi > t\} = \exp(-ct^\beta)$, $\beta > 1$. In the beginning we have only one particle but birth and diffusion processes lead to the “occupation of the space”. This problem of the quantitative description of this phenomena is similar to the famous problem KPP (Kolmogorov-Petrovski-Piskunov).

It is not very difficult to show that the boundary of “occupied” region in moment $t$ is given by a sphere

$$S_t = \{x : |x| \leq t \ln^{1/\beta} t\}$$

in the following way: if $|x| > t \ln^{1/\beta+\epsilon} t$, then $c(t, x) \overset{P}{\rightarrow} 0$ and if $|x| < t \ln^{1/\beta-\epsilon} t$, then $c(t, x) \overset{P}{\rightarrow} \infty$.

However, the set $\sigma_t$ is extremely nonuniformly occupied, as is a typical effect of intermittency.

**Theorem 12.** For every $t > 0$, $\epsilon > 0$ it’s possible to find (random) points $x_1(t, w), \ldots, x_k(t, w)$, $k < \ln t$, such that

$$\sum_{x \in \mathbb{Z}^d} c(t, x) \geq (1 - \epsilon) \sum_{i=1}^k c(t, x_i). \quad (18)$$

It is likely that $k = k(t, w)$ is bounded in probability. This theorem shows that it is necessary to be very careful in the applications of the results on the averaging description of the propagation front of concentration in random media. The field of concentration inside the front has an extremely non-uniform structure.
§ 5. On the Basic States in the Anderson Model.
Precision of the Asymptotical Formulas for “Lifshitz Tails”

The limit theorems for the boundary part of the spectrum, that is for the basic states will be considered in finite, but big volume $V$ when $V \to \infty$. The integral density of states $N(A)$ can be studied in the framework of the same procedure.

Let $S_N^d = [-N,N]^d$, where points $N,-N$ are identified, be the $d$ dimensional lattice torus of the volume $V_N = (2N)^d$ and $H = \Delta + \xi(x)$ be the operator of the tight-binding Anderson model. This means that $\xi(x), x \in S_N^d$ is an i.i.d.r.v. We will consider a typical one for the theory of “Lifshitz tails” case

$$P\{\xi_x > t\} = \exp\{-ct^\beta\}, \ t > 0, \ \beta > 0.$$

We are interested in the structure of higher eigenvalues and corresponding eigenfunctions. It’s easy to understand that they are closely connected with the higher peaks of the potential $\xi(x), x \in S_N^d$. Consider the two variational series:

$$\xi_N^{(1)} > \xi_N^{(2)} > \ldots > \xi_N^{(V)}$$
$$\lambda_N^{(1)} > \lambda_N^{(2)} > \ldots > \lambda_N^{(V)}.$$

The limit distribution of any fixed number $k$ of the first r.v. in the $\xi$-series is described by the Weibull’s type low:

$$P \left\{ \frac{\xi_N^{(1)} - \xi_N^{(2)}}{\frac{1}{p} \ln^{(1/p-1)} V_N} \in (x_1 + dx_1), \frac{\xi_N^{(2)} - \xi_N^{(3)}}{\frac{1}{p} \ln^{(1/p-1)} V_N} \in (x_2 + dx_2), \ldots, \frac{\xi_N^{(k)} - \ln^{1/p} V_N}{\frac{1}{p} \ln^{(1/p-1)} V_N} \in (x_k + dx_k) \right\}$$

$$\to_{N \to \infty} P_k(x_1, \ldots, x_k) = \exp(-kx_1 - 2kx_2 - \ldots - kx_k - e^{-x_k}), \ x_1, \ldots, x_k \geq 0.$$

It seems very reasonable, that the corresponding (or near) formulas are valid for $\lambda_N^{(i)}, \ i = 1, 2, \ldots, k$. In some sense it is true.

**Theorem 13.** For some positive $A_N^{(1)}, \ldots, A_N^{(k)}, B_N$,

$$P \left\{ \frac{\lambda_N^{(1)} - \lambda_N^{(2)} - A_N^{(1)}}{B_N} > x_1, \frac{\lambda_N^{(2)} - \lambda_N^{(3)} - A_N^{(2)}}{B_N} > x_2, \ldots, \frac{\lambda_N^{(k)} - A_N^{(k)}}{B_N} > x_k \right\}$$

$$\to_{N \to \infty} \int_{x_1}^{\infty} \ldots \int_{x_k}^{\infty} \exp(-y_1 - 2y_2 - \ldots - ky_k - \exp(-y_k)) dy_1 \ldots dy_k,$$

$x_1, \ldots, x_k > 0$.

The structure of normalizing constants, however, depends on $\beta$. There exist many bifurcations with respect to $\beta$. The nature of these bifurcations is very simple. It is obvious that only $\xi$-peaks which have an order $O(\ln^{1/\beta} V)$, for example, bigger then $(1 - \varepsilon) \ln^{1/\beta} V$ can contribute to the initial part of the $\lambda$-series. But $\#\{x \in S_N^d : \xi(x) > (1 - \varepsilon) \ln^{1/\beta} V\} = O(V^{\delta(\varepsilon)}); \ \delta(\varepsilon) \to 0, \ \varepsilon \to 0$ and the
distances between these $\xi$-peaks have an order $N^{1-\delta_1(\varepsilon)}$; $\delta_1(\varepsilon) \to 0$, $\varepsilon \to 0$. The interaction between peaks is very small and any of them as shown by standard perturbation calculations gives an eigenvalue

$$
\lambda(x_0) = \xi(x_0) - \frac{c_0}{\xi(x_0)} + c_1 \sum_{|x-x'|=1} \frac{\xi(x')}{\xi^2(x_0)} + \mathcal{O}\left(\frac{1}{\xi^3(x_0)}\right). \quad (21)
$$

If $p < 2$, then the second and all other terms are small enough with respect to the "gaps" between neighboring $\xi^{(i)}$ and we can use formula (19), changing $\xi^{(i)}$ by $\lambda^{(i)}$. If $2 \leq p < 3$, then the second term of expansion (21) is essential. It is necessary to slightly change constant $A^{(i)}$. However, as in the case $p < 2$, we have the correspondence $\xi^{(1)} \leftrightarrow \lambda^{(1)}$, $\xi^{(2)} \leftrightarrow \lambda^{(2)}$, ..., $\xi^{(k)} \leftrightarrow \lambda^{(k)}$. If $p \geq 3$, then the gaps between $\xi^{(1)} - \xi^{(2)}$, ... are smaller than $\frac{1}{\xi^3(x_0)}$, this correspondence is destroyed and all normalized constants are new. It is not possible to write out explicit formulas for $A^{(i)}$, $B_N$ as functions of $\beta$ and dimension $d$.

The same analysis applies to the problem of "Lifshitz tails" as to the high energy asymptotic of $N^*(\lambda) = 1 - N(\lambda) = \lim_{\nu \to \infty} \frac{1}{\nu} \sum_{\lambda_N > \lambda} 1$. It is well known that

$$
P\{\xi > \lambda + 2d\} < N^*(\lambda) < P\{\xi > \lambda - 2d\}.
$$

(Of course, $2d = ||A||_\rho$). It follows from these estimations that

$$
- \ln N^*(\lambda) \sim \lambda^\beta.
$$

What is the more precise form of this asymptotics? The answer depends primarily on $\beta$ and contains information on the structure of corresponding eigenfunctions.

**Theorem 14.**

a) If $p < 2$, then

$$
- \ln N^*(\lambda) = \lambda^\beta + \lambda^\beta + \mathcal{O}(1). \quad (22)
$$

b) If $p = 2$ (gaussian case), then

$$
- \ln N^*(\lambda) = \lambda^\beta + c_1(d) + \mathcal{O}(1). \quad (22)
$$

c) If $2 < p < 3$

$$
- \ln N^*(\lambda) = \lambda^\beta + c_2 \lambda^\beta - 2 + \mathcal{O}(1). \quad (23)
$$

d) If $3 < p < 4$

$$
- \ln N^*(\lambda) = \lambda^\beta + c_2 \lambda^\beta - 2 + c_3 \lambda^{(\beta-2)-\frac{2}{\beta-1}} (1 + \mathcal{O}(1)) \quad (24)
$$

and so on.

There is no room here to discuss other types of bifurcations of $N(\lambda)$ in the case of "double exponential" tails of the distribution $\xi(\cdot)$. The situation here is similar to the results of Theorems 10 and 11.
References


