1. Introduction

The topological entropy of a continuous dynamical system is now well established as an important invariant of the system. It was first defined by Adler, Konheim, and McAndrew [AKM] in 1965 using open coverings. Nowadays it is convenient to think of the topological entropy as a limit of the number of the distinct orbits of a given length which can be obtained with a fixed small precision. Thus, it measures the orbit growth of the system.

More precisely, suppose that \( f : M \to M \) is a continuous self-mapping of the compact metric space \( M \) with metric \( d \). Given a positive integer \( n \), and a small real number \( \delta > 0 \), we say that a set \( E \) is an \((n, \delta)\)-separated set if, for any \( x \neq y \in E \) there is a \( j \in [0, n) \) such that \( d(f^j x, f^j y) > \delta \). Let \( r(n, \delta, f) \) be the maximum cardinality of an \((n, \delta)\)-separated set in \( M \). Let

\[
\begin{align*}
    h(\delta, f) &= \limsup_{n \to \infty} -\frac{1}{n} \log r(n, \delta, f) \\
    h(f) &= \lim_{\delta \to 0} h(\delta, f) = \sup_{\delta > 0} h(\delta, f).
\end{align*}
\]

This is the topological entropy of \( f \). The most interesting properties of \( h(f) \) arise from its relation to set of invariant probability measures of \( f \). Let us denote this set by \( \mathcal{M}(f) \), and recall that it is a compact metrizable set. Given \( \mu \in \mathcal{M}(f) \), the measure-theoretic or metric entropy, \( h_\mu(f) \), is defined as follows. For a finite Borel measurable partition \( \alpha \), let

\[
H_\mu(\alpha) = -\sum_{A \in \alpha} \mu(A) \log \mu(A).
\]

Given two finite partitions \( \alpha, \beta \), set \( \alpha \vee \beta = \{ A \cap B : A \in \alpha, B \in \beta \} \). Then, set

\[
h_\mu(\alpha, f) = \lim_{n \to \infty} -\frac{1}{n} H_\mu(\bigvee_{i=0}^{n-1} f^{-i}(\alpha))
\]

Adler recently pointed out to us that topological entropy generalizes Shannon's notion of channel capacity (see [SW, p.7])
and
\[ h_\mu(f) = \sup_\alpha h_\mu(\alpha, f) = \lim_{\text{diam } \alpha \to 0} h_\mu(\alpha, f). \]

The following properties hold:

1. \( h(f^n) = nh(f), \quad h_\mu(f^n) = nh_\mu(f) \) for \( n \in \mathbb{Z}^+ \)
2. \( h(f^t) = t h(f^1) \) if \( \{f^t\}_{t \in \mathbb{R}} \) is a continuous flow.
3. \( h(f) = \sup_{\mu \in \mathcal{M}(f)} h_\mu(f) \)
4. \( h(f) = h(g) \) if \( f \) is topologically conjugate to \( g \); i.e., there is a homeomorphism \( \phi \) with \( \phi f \phi^{-1} = g \).
5. \( h(f) \) can be computed using \( (n, \delta) \)-separated subsets of sets of large measure for various \( \mu \in \mathcal{M}(f) \) (see [N1]). That is,
\[
\begin{align*}
   h(f) &= \sup_{\mu \in \mathcal{M}(f)} \lim_{\sigma \to 1} \inf_{\mu(\Lambda) > \sigma} \lim_{\delta \to 0} \bar{r}(\delta, \Lambda) \\
   \text{where} \\
   \bar{r}(\delta, \Lambda) &= \lim_{n \to \infty} \sup_{n} \frac{1}{n} \log \bar{r}(\delta, n, \Lambda)
\end{align*}
\]
and \( \bar{r}(\delta, n, \Lambda) \) is the maximal cardinality of an \((n, \delta)\)-separated subset of \( \Lambda \).

The relationship between topological and metric entropies (statement 3 above) is a combination of the work of Goodman, Goodwyn, and Dinaburg. It is referred to as the Variational Principle for Topological Entropy. An elegant proof has been given by Misiurewicz (see [DGS], pp. 140–146 for a more general result). Statement 4 above states that \( h(f) \) is a topological conjugacy invariant of \( f \). It is generally not a complete invariant of topological conjugacy. However, for certain important systems it is close to being complete. For instance, Adler and Marcus [AM] have shown that two mixing subshifts of finite type with the same topological entropy are almost topologically conjugate in the sense that each is a boundedly finite to one factor of a third mixing subshift of finite type. Also, Milnor and Thurston [MT] have shown that a piecewise monotone continuous map \( f \) of an interval with positive topological entropy is semi-conjugate in a simple way to a piecewise linear map \( g \) with the same number of turning points as \( f \) and such that the slope of each monotone piece of \( g \) has absolute value equal to \( e^{h(f)} \).

It was known quite early that some form of regularity affected the finiteness of the topological entropy. Embedding a sequence of larger and larger shift automorphisms topologically in a homeomorphism of the two-sphere shows that homeomorphisms need not have finite topological entropy. However, Kushnirenko proved that every \( C^1 \) self-map of a compact manifold has finite topological entropy. A natural question arises: When are there measures \( \mu \) for which \( h_\mu(f) = h(f)? \) Misiurewicz was the first to construct examples of \( C^k \) diffeomorphisms of compact manifolds with no measure of maximal entropy. His first examples were on manifolds of dimension greater than three, but now it is known that such examples arise in small perturbations of diffeomorphisms having a degenerate homoclinic orbit (an intersection of stable and unstable manifolds of a
hyperbolic saddle point with infinite order contact) even on the two dimensional sphere. His construction worked for every finite \( k \), but it failed in the \( C^\infty \) case. The question of the existence of measures of maximal entropy was quite important. One consequence of the results described here is that for every \( C^\infty \) self-map \( f \) of a compact smooth manifold, the function \( \mu \to h_\mu(f) \) is uppersemicontinuous on \( M(f) \). Hence, \( f \) does indeed possess measures of maximal entropy.

In this article we shall survey several recent results about topological and metric entropy, particularly as they relate to smooth systems. We view the results described here as part of the natural evolution of certain topological aspects of the qualitative theory of dynamical systems following the rich development in the sixties due, in large part, to Anosov, Sinai, and Smale. The recent developments on quadratic mappings due to Carleson and Benedicts may be viewed as part of the evolution of certain quantitative aspects of this theory. In the case of uniformly hyperbolic dynamics, these two types of aspects come together beautifully in the theory of equilibrium states as described, for example, in [B2]. It is natural to search for a generalization of the Equilibrium State Theory which encompasses all of these results.

2. Entropy and Volume

To motivate the results, we first consider the case of mappings of the interval. Let \( M \) be the unit interval and let \( f : M \to M \) be a continuous map with finitely many turning points. Let \( \log^+(x) = \max(\log x, 0) \).

Let \( \ell(f) = \text{length of image of } f \) with multiplicities:

\[
\ell(f) = \int_M |f'(x)| \, dx
\]

Theorem 1 (Misiurewicz-Szlenk [MS]). With \( f \) as above, the following results hold.

1. \( h(f) = \lim_{n \to \infty} \frac{1}{n} \log^+ \ell(f^n) \).
2. \( \mu \to h_\mu(f) \) is uppersemicontinuous on \( M(f) \).
3. \( f \to h(f) \) is uppersemicontinuous for \( f \) \( C^1 \) where one perturbs in the \( C^1 \) topology keeping the same number of turning points.
4. \( f \to h(f) \) is lowersemicontinuous for certain \( f \).

Theorem 2 (Misiurewicz [M2]). The map \( f \to h(f) \) is lowersemicontinuous for all \( C^0 \) \( f \).

Corollary 3. The map \( f \to h(f) \) is continuous for \( f \) in the set of \( C^1 \) maps with a uniformly bounded number of turning points.

This last corollary was also proved by Milnor and Thurston [MT].

We consider a direct generalization of the preceding results. A clue as to how to proceed comes from work of Margulis in the sixties on the geodesic flow on compact negatively curved manifolds. Following his work, it became known that
the topological entropy of the time-one map equals the maximum volume growth rate of compact disks in the unstable manifolds.

Let $D^k$ be the unit $k$-disk in $\mathbb{R}^k$, and let $M$ be a $C^\infty$ manifold. Let $C^r(M,M)$ be the space of $C^r$ self maps of $M$, and let $\mathcal{D}^r(M,M)$ be the space of $C^r$ diffeomorphisms of $M$ where $r$ is an integer greater than 1.

A $C^r$ $k$-disk in $M$ is a $C^r$ map $\gamma : D^k \to M$.

Define the $k$-volume of $\gamma$ by

$$|\gamma|_k = \int_{D^k} |A^k T\gamma| d\lambda$$

where $d\lambda$ is Lebesgue measure on $D^k$, and $A^k T\gamma$ is the $k$-th exterior power of the derivative $T\gamma$

For $f \in C^r(M,M)$, let $A$ be a compact $f$-invariant set, and let $U$ be a compact neighborhood of $A$. Given a positive integer $n$, set $W^s(n, U) = \cap_{0 \leq j < n} f^{-j}(U)$. This is just the set of points whose iterates from time 0 through $n - 1$ remain in $U$. For a $k$-disk $\gamma$ in $U$, set

$$G_k(\gamma, f, U) = \limsup_{n \to \infty} \frac{1}{n} \log^+ (|f^{n-1} \circ \gamma|_{\gamma^{-1}(W^s(n, U))} |_k).$$

This is the volume growth of the $f^{n-1}$-st iterate of the part of $\gamma$ which remains in $U$ from time 0 through time $n - 1$.

A collection $\mathcal{A}$ of $k$-disks in $U$ with $1 \leq k \leq \dim M$, is called ample for $A$ if it contains a subcollection $\mathcal{A}_1$ for which $\exists K > 0$ such that

1. $K^{-1} \leq |D_x \gamma(v)| \leq K$ and $|D^2_x \gamma(v,v)| \leq K$

2. For $x \in A$, and for each $k$-dimensional subspace $H$ of $T_x M$, there exists a sequence of $k$-disks $\gamma_1, \gamma_2, \ldots, \in \mathcal{A}_1$ whose tangent spaces at $\gamma_i(0)$ approach $H$ in the Grassmann sense as $i \to \infty$.

If $M$ is complex analytic with a hermitian metric, $A$ and $U$ are as above, and $\mathcal{A}$ is a collection of holomorphic disks in $U$, then $\mathcal{A}$ is holomorphically ample or $h$-ample if, in condition 2 above, $H$ is assumed to be a complex subspace of the holomorphic tangent space of $T_x M$.

Clearly the family of all disks through points in $A$ is ample for $A$. If $M = \mathbb{R}^N$, with the usual metric, then the collection $\mathcal{A}$ of affine disks through points in $A$ is ample.

Theorem 4 gives an upper bound for entropy in terms of volume growth rates of smooth disks. An earlier upper bound in terms of the average (relative to Lebesgue measure) of the maximum growth of the norms of the exterior powers of the derivative had been obtained by Przytycki [P] for diffeomorphisms.
Theorem 4 [N1]. Suppose \( f \in C^r(M,M) \), \( A \) is a compact invariant set, \( U \) is a compact neighborhood of \( A \), and \( \mathcal{A} \) is an ample family of \( C^r \) disks for \( A \). If \( f \) and \( M \) are complex analytic assume that \( \mathcal{A} \) is h-ample. Then,

\[
h(f \mid A) \leq \sup_{\gamma \in \mathcal{A}} G(\gamma, f, U).
\]

If \( f \in \mathcal{D}^r(M, M) \), then

\[
h(f \mid A) \leq \sup_{\dim \gamma < \dim M} G(\gamma, f, U).
\]

In particular, for \( f \in \mathcal{D}^r(M^2) \), \( h(f \mid A) \leq \sup_{\text{curves } \gamma} G(\gamma, f, U) \).

Using Theorem 4, one can give simple proofs of the following results.

Theorem 5. 1. Let \( f : \mathbb{R}^N \rightarrow \mathbb{R}^N \) be a polynomial map with coordinate functions of degree \( \leq d \); i.e., for \( x \in \mathbb{R}^N \), \( f(x) = (f_1(x), f_2(x), \ldots, f_N(x)) \) with \( f_i(x) \) a polynomial in \( x \) of degree \( \leq d \). If \( A \) is a compact invariant set for \( f \), then

\[
h(f \mid A) \leq N \log d \quad \text{(Gromov)}.
\]

2. If \( f : S^2 \rightarrow S^2 \) is a rational map of degree \( d \), then

\[
h(f) \leq \log d \quad \text{(Gromov — Ljubich)}.
\]

3. If \( f : \mathcal{P}^N(\mathbb{C}) \rightarrow \mathcal{P}^N(\mathbb{C}) \) is globally defined and holomorphic, then \( h(f) \leq \log(\text{topological degree}) \quad \text{(Gromov)} \).

S. Friedland [F] has obtained results on volume growth in quasi-projective varieties which generalize Theorem 5.1.

It is well-known (Misiurewicz-Przytycki [MP]) that if \( M \) is compact and \( f : M \rightarrow M \) is \( C^1 \), then \( h(f) \geq \log(\text{topological degree}) \).

So, Theorems 5.2 and 5.3 above are equalities.

From now on, we assume that \( M \) is a compact \( C^\infty \) manifold.

The above theorems give an upper estimate of \( h(f) \) in terms of volume growth rates of disks. For the lower estimate we have

Theorem 6 (Yomdin [Y1]). For \( f \in C^\infty(M, M) \), and any \( C^\infty \) disk \( \gamma \) in \( M \),

\[
h(f) \geq G(\gamma, f).
\]

Yomdin's results can be used to give a proof of a generalization of the Shub entropy conjecture in the \( C^\infty \) case. To recall this conjecture, first define the homology growth of a map \( f \), \( HG(f) \), to be \( \limsup_{n \to \infty} \frac{1}{n} \log |f^n_*| \) where \( f_* : H_*(M, \mathbb{R}) \rightarrow H_*(M, \mathbb{R}) \) is the induced map on the direct sum of the real homology groups of \( f \) (given any norm). The entropy conjecture states that for a \( C^1 \) diffeomorphism \( f \) of the compact manifold \( M \), one has \( h(f) \geq HG(f) \). Of course, \( HG(f) \) is the same as the maximum logarithm of the absolute values of the eigenvalues of \( f_* : H_*(M, \mathbb{R}) \rightarrow H_*(M, \mathbb{R}) \). Yomdin's results show that this holds for arbitrary \( C^\infty \) maps.
Corollary 7 (Yomdin) ($C^\infty$ Entropy Conjecture). If $f \in C^\infty(M,M)$, then $h(f) \geq HG(f)$.

We note that the Entropy Conjecture fails in general for piece-wise linear homeomorphisms although it is true for "typical" piecewise linear maps. There is a large literature on various cases of the entropy conjecture (see [FS]). The general conjecture is still unproved for $f \in \mathcal{D}^r(M,M)$ with $1 \leq r < \infty$.

The next result states that, for positive $h(f)$, there always exist disks $\gamma$ for which $G(\gamma, f)$ assumes the maximum value. In addition, it can be shown that there are such disks for which $G(\gamma, f)$ is actually a limit and not just a lim sup.

**Theorem 8.** For $f \in C^\infty(M,M)$, $\mathcal{A}$, an ample family of $C^\infty$ disks, we have

$$h(f) = \sup_{\gamma \in \mathcal{A}} G(\gamma, f) = \max_{\gamma \in \mathcal{A}} G(\gamma, f).$$

In general, the disks $\gamma$ for which $G(\gamma, f) = h(f)$ are not easily identifiable. However, for an area decreasing self-embedding of a surface with boundary, the entropy is just the growth rate of the length of the boundary.

**Theorem 9 [N2].** For $f \in D^\infty(M^2)$, $\partial M^2 \neq \emptyset$, $f$ area decreasing, we have

$$h(f) = G(\partial M^2, f).$$

### 3. Continuity Properties of Entropy

The methods used in the proofs of the above results concerning volume growth and entropy have local analogs by which we mean that one considers the growth rates of the cardinalities of $(n, \delta)$—separated sets or of the volumes of disks which remain in small neighborhoods of the orbits of given points. These can be combined with general results estimating the defect in uppersemicontinuity of both topological and metric entropy to obtain various continuity properties of entropy for $C^\infty$ systems.

We begin with a description of the so-called local entropy of a dynamical system $f : M \to M$.

Given $A \subset M$, $x \in A$, $n \in \mathbb{Z}^+$, $\varepsilon > 0$, set

$$W_x(n, \varepsilon, A) = \{ y \in A : d(f^j y, f^j x) < \varepsilon \ \forall j \in [0, n) \},$$

$$r(n, \delta, \varepsilon, A) = \sup_{x \in A} \max \{ \text{card } E : E \subset W_x(n, \varepsilon, A), E \text{ is } (n, \delta) \text{—separated} \},$$

$$r(\varepsilon, A) = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \delta, \varepsilon, A).$$

If $\mu \in \mathcal{M}(f)$, let

$$h_{\mu loc}(\varepsilon, f) = \lim_{\sigma \to 1} \inf_{\mu(A) > \sigma} r(\varepsilon, A),$$

and set,

$$h_{\mu loc}(\varepsilon, f) = \sup_{\mu} h_{\mu loc}(\varepsilon, f).$$
The quantity $h_{\text{loc}}(\varepsilon,f)$ is called the $\varepsilon$--local entropy of $f$, and $h_{\mu,\text{loc}}(\varepsilon,f)$ is called the $\varepsilon$--local entropy of $f$ relative to $\mu$.

The next theorem states that $h_{\text{loc}}(\varepsilon,f)$ gives an upper bound for the difference $h(f) - h(\varepsilon,f)$ while $h_{\mu,\text{loc}}(\varepsilon,f)$ gives an upper bound for the difference $h_{\mu}(f) - h_{\mu}(\beta,f)$ for any partition $\beta$ with diameter less than $\varepsilon$. Earlier estimates of these differences were given by Bowen in [B].

**Theorem 10** [N2]. For any continuous self map $f$ of the compact metric space $M$, and $\varepsilon > 0$,

1. $h(f) \leq h(\varepsilon,f) + h_{\text{loc}}(\varepsilon,f)$
2. If $\mu \in \mathcal{M}(f)$ and $\beta$ is a finite Borel partition with diam $\beta < \varepsilon$, then $h_{\mu}(f) \leq h_{\mu}(\beta,f) + h_{\mu,\text{loc}}(\varepsilon,f)$

Next we consider local volume growth.

Let $W_x(n,\varepsilon) = W_x(n,\varepsilon,M)$.

For a disk $\gamma$, set

$$G_{\text{loc}}(\varepsilon,\gamma) = \lim_{n \to \infty} \frac{1}{n} \log^+ \sup_{x \in M} | f^{n-1} \circ \gamma | \gamma^{-1}(W_x(n,\varepsilon)) |$$

and

$$G_{\text{loc}}(\varepsilon,f) = \sup_{\gamma} G_{\text{loc}}(\varepsilon,\gamma)$$

**Theorem 11** [N2]. For $f \in C^r(M,M)$, $r > 1$,

$$h_{\text{loc}}(\varepsilon,f) \leq G_{\text{loc}}(2\varepsilon,f)$$

**Theorem 12** (Yomdin[Y1]). For $f \in C^\infty(M,M)$,

$$\lim_{\varepsilon \to 0} G_{\text{loc}}(\varepsilon,f) = 0$$

From Theorems 10, 11, 12 with some elementary arguments (see [N2]) we have

**Theorem 13.** 1. For $f \in C^\infty(M,M)$, $\mu \to h_{\mu}(f)$ is uppersemicontinuous
2. $f \to h(f)$ is uppersemicontinuous on $C^\infty(M,M)$.

Yomdin [Y1] has an independent proof of Theorem 13.2.

In general, $f \to h(f)$ is not lowersemicontinuous, but Katok has proved that this does hold on surfaces.

**Theorem 14** (Katok). $f \to h(f)$ is lowersemicontinuous on $C^2(M^2)$

**Corollary 15.** $f \to h(f)$ is continuous on $C^\infty(M^2)$. 
Question 16. Is the preceding map Holder continuous?

Answer: No

Yomdin [Y2] has examples of curves \( \{ f_t \} \) of real analytic maps on \( S^2, t \in [-\varepsilon, \varepsilon] \) with \( h(f_t) = 0 \) for \( t \in [-\varepsilon, 0] \), and for \( t \in (0, \varepsilon) \),

\[
h(f_t) - h(f_0) > C \frac{\log |\log t|}{\log t}.
\]

In view of Corollary 15, we wish to point out some analogies between the entropy map \( f \to h(f) \) on \( D^\omega(M^2) \) and the rotation number map \( f \to g(f) \in \mathbb{R}/\mathbb{Z} \) on \( \text{Homeo}^+(S^1) \), the set of orientation preserving homeomorphisms of the circle. Both \( h(f) \) and \( g(f) \) are topological invariants which depend continuously on \( f \). Moreover, \( g(f) \) is rational iff \( f \) has periodic points for \( f \in \text{Homeo}^+(S^1) \) while (as proved by Katok) \( h(f) \) is positive iff \( f \) has transverse homoclinic points for \( f \in D^\omega(M^2) \).

Problems

1. (Monotonicity of entropy) For \( f_r(x) = r - x^2 \), the function \( r \to h(f_r) \) is monotone increasing (Douady and Hubbard). What about \( r \to h(f_{r,b}) \) for fixed \( b \) with \( f_{r,b}(x, y) = (r - x^2 + by, x) \)?

2. Let \( \mathcal{M}_{\text{max}}(f) \) denote the set of measures of maximal entropy for a mapping \( f \). As a consequence of Theorem 13, for any \( C^\omega f \), \( \mathcal{M}_{\text{max}}(f) \neq \emptyset \). For \( f \in D^\omega(M^2) \) with \( h(f) > 0 \), is \( \mathcal{M}_{\text{max}}(f) \) a finite dimensional simplex?

Related to this problem, Hofbauer ([H]) has developed an interesting theory concerning piecewise monotone mappings of an interval with finitely many monotone continuous pieces. His theory can be described by introducing the notion of a zero-entropy set (0-entropy set). Hofbauer calls these small sets.

Let \( f : X \to X \) be a Borel automorphism of a standard Borel space. A 0-entropy set is an \( f \)-invariant subset \( X_1 \subset X \) such that for any ergodic \( \mu \in \mathcal{M}(f) \), with \( \mu(X_1) = 1 \), we have \( h_\mu(f) = 0 \). By convention, if \( \mathcal{M}(f) = \emptyset \), then, every invariant subset of \( X \) is a 0-entropy set. Periodic orbits are simple 0-entropy sets as are stable manifolds of hyperbolic periodic orbits in the smooth setting. There is a natural notion of isomorphism mod 0-entropy: Borel automorphisms \( (f, X), (g, Y) \) are isomorphic mod 0-entropy if there are 0-entropy sets \( X_1 \subset X, Y_1 \subset Y \), and a Borel isomorphism \( \phi : X \setminus X_1 \to Y \setminus Y_1 \), with \( \phi \circ f = \phi \circ g \). We say two Borel endomorphisms \( f : X \to X, g : Y \to Y \) are isomorphic mod 0-entropy if their natural extensions are isomorphic mod 0-entropy. Finally, we say that the Borel endomorphism \( (f, X) \) is Markov mod 0-entropy if it is isomorphic mod 0-entropy to a finite or countable state topological Markov chain \((\sigma, \Sigma_\Lambda)\). We will say that a measure preserving endomorphism \((X, f, \mu)\) is essentially Markov if its natural extension \((\hat{X}, \hat{f}, \hat{\mu})\) is isomorphic to a Markov process. In this case we also say that \( \mu \) is essentially Markov.

Theorem 17 (Hofbauer). Let \( f : I \to I \) be a piecewise monotone map of the interval. Then, \((f, I)\) is Markov mod 0-entropy. Moreover, there are only finitely many ergodic measures of maximal entropy and each is essentially Markov.
Moving to general $C^{\infty}$ maps of an interval with positive topological entropy, we can prove that there are at most a countable number of ergodic measures of maximal entropy, and that each is essentially Markov.

In dimension greater than 1, it is not always true that positive entropy implies that $\mathcal{M}_{\text{max}}(f)$ is a finite dimensional simplex: take the direct product of the identity transformation and any transformation with positive entropy. However, it is possible that, generically, i.e. for elements of a residual set of diffeomorphisms, $\mathcal{M}_{\text{max}}(f)$ is a finite dimensional simplex.

3. For $f \in \mathcal{G}^r(M^2)$, $h(f) = 0$, can $f$ be $C^r$ perturbed to be Morse-Smale? This is true if the limit set of $f$ is finite and hyperbolic $[\text{MaP}]$, but not even known if the non-wandering set is finite.

4. For $f \in \mathcal{G}^{\infty}(M^2)$, let $\chi^+(x)$ denote the positive characteristic exponent of $x$ (defined for a total probability set of $x$). Let $\phi(x) = -\chi^+(x)$. Then, $\phi$ is bounded and Borel measurable. Define

$$P(\phi) = \sup_{\mu \in \mathcal{M}(f)} h_\mu(f) + \int \phi d\mu.$$ 

Is there always a $\mu_0$ such that

$$P(\phi) = h_{\mu_0}(f) + \int \phi d\mu_0.$$ 

This is true for continuous $\phi$. Such $\mu_0$s are called $\phi$-equilibrium states. For a hyperbolic attractor $A$ and any $\mu$ supported in the basin of $A$ and absolutely continuous with respect to Lebesgue measure, it is known that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f^k_*(\mu)$$

is the unique $\phi$-equilibrium state on $A$ (Ruelle). In general, one would expect $\phi$-equilibrium states to be related to weak limits of the measures $\{\frac{1}{n} \sum_{k=0}^{n-1} f^k_*(\mu)\}$. In this connection, Pesin and Sinai have shown that the weak limits of the iterates of Lebesgue measure have absolutely continuous densities along unstable leaves for partially hyperbolic attractors $[\text{PS}]$.

References


[Y2] Y. Yomdin: Preprint