The Work of Pierre-Louis Lions

S. R. S. VARADHAN

Courant Institute, New York University
251 Mercer Street, New York, NY 10012, USA

Pierre-Louis Lions has made unique contributions over the last fifteen years to mathematics. His contributions cover a variety of areas, from probability theory to partial differential equations (PDEs). Within the PDE area he has done several beautiful things in nonlinear equations. The choice of his problems has always been motivated by applications. Many of the problems in physics, engineering and economics when formulated in mathematical terms lead to nonlinear PDEs; these problems are often very hard. The nonlinearity makes each equation different. The work of Lions is important because he has developed techniques that, with variations, can be applied to classes of such problems. To say that something is nonlinear does not mean much; in fact it could even be linear. The entire class of nonlinear PDEs is therefore very extensive and one does not expect an all-inclusive theory. On the other hand, one does not want to treat each example differently and have a collection of unrelated techniques. It is thus extremely important to identify large classes that admit a unified treatment.

In dealing with nonlinear PDEs one has to allow for nonclassical or nonsmooth solutions. Unlike the linear case one cannot use the theory of distributions to define the notion of a weak solution. One has to invent the appropriate notion of a generalized solution and hope that this will cover a wide class and be sufficient to yield a complete theory of existence, uniqueness, and stability for the class.

Due to the very limited time that is available, I shall focus on three areas within nonlinear PDE where Lions has made major contributions. The first is the so called "viscosity method". This development is a long story that started with some work in collaboration with Crandall. Over many years, in partial collaboration with others (besides Crandall, Evans and Ishii), Lions has developed the method, which is applicable to the large class of nonlinear PDEs known as fully nonlinear second order degenerate elliptic PDEs. The class contains very many important subclasses that arise in different contexts.

By solving a nonlinear PDE one is trying to solve an equation involving an unknown function and its derivatives. Let u be a function in a region G in some R^n and let $Du, D^2u, \ldots, D^ku$ be its derivatives of order up to k. A nonlinear PDE is an equation of the form

$$F[x, u(x), (Du)(x), (D^2u)(x), \ldots, (D^k(u)(x))] = 0 \quad \text{in} \quad G$$

with some boundary conditions on ∂G. Such a PDE is said to be nonlinear and of order k. The viscosity method applies in cases where $k = 2$ and $F(x, u, p, H)$

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has certain monotonicity properties in the arguments u and H. More precisely, it is nondecreasing in u and nonincreasing in H. Here u is a scalar and H is a symmetric matrix of size $n \times n$ with the natural partial ordering for symmetric matrices.

Some of the many examples of such functions are described below.

Linear elliptic equations:

$$-\sum_{i,j} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + f(x) = 0$$

where the matrix $a^{ij}(x)$ is uniformly positive definite.

In this case the function F is given by

$$F(x,u,p,H) = -\text{Trace}[a(x)H] + f(x)\,.$$

First order equations:

$$f(x,u(x),(Du)(x)) = 0$$

These include Hamilton-Jacobi equations where it all started. One added a term of the form $\epsilon\Delta$ to the equation and constructed the solution in the limit as ϵ went to zero. The theory owes its name to its early origins.

If one has a family F_α of such functions one can generate a new one by defining

$$F = \sup_\alpha F_\alpha\,.$$

If one has a two-parameter family $F_{\alpha\beta}$ of such functions one can generate a new one by defining

$$F = \sup_\alpha \inf_\beta F_{\alpha\beta}\,.$$

Such examples arise naturally in control theory and game theory and are referred to as Hamilton-Jacobi-Bellman and Isaacs equations.

In order to understand the notion of a generalized solution it is convenient to talk about supersolutions and subsolutions. Suppose u is a subsolution, i.e.

$$F(x,u(x),(Du)(x),(D^2u(x))) \le 0$$

and we have another function ϕ, which is smooth, such that $u-\phi$ has a maximum at some point $\hat{x}$. Then by calculus $Du(\hat{x}) = D\phi(\hat{x})$ and $D^2(u)(\hat{x}) \le D^2(\phi)(\hat{x})$. From the monotonicity properties of F it follows that

$$F(\hat{x},u(\hat{x}),(Du)(\hat{x}),(D^2u(\hat{x}))) \ge F(\hat{x},u(\hat{x}),(D\phi)(\hat{x}),(D^2\phi(\hat{x})))\,.$$

Therefore

$$F(\hat{x},u(\hat{x}),(D\phi)(\hat{x}),(D^2\phi(\hat{x}))) \le 0.$$

The last inequality makes sense without any smoothness assumption on u. We can try to define a nonsmooth subsolution as a u that satisfies the above for arbitrary smooth ϕ and $\hat{x}$ provided $u - \phi$ has a maximum at $\hat{x}$. The definition of a supersolution is similar, and a solution is one that is simultaneously a super and a subsolution.

Let us consider first a Dirichlet boundary value problem where we want to find a u that solves our equation and has boundary value zero.

The main step is to establish the key comparison theorem (with a long history that began with the work of Crandall and Lions and saw an important contribution from Jensen) that if u is a subsolution and if v is a supersolution in a bounded domain G and if $u \leq v$ on the boundary ∂G then $u \leq v$ in $G \cup \partial G$. This requires some mild regularity conditions on F as well as some nondegeneracy conditions. After all, we have not ruled out $F \equiv 0$. Once such conditions are imposed one can establish the key comparison theorem. From this point on, the theory proceeds in a way similar to the classical Perron's method for solving the Dirichlet problem. Assuming that there is at least one subsolution $\bar{u}$ and at least one supersolution $\bar{v}$ with the the given boundary value, one establishes that

$$W(x) = \sup\{w(x) : \bar{u} \leq w \leq \bar{v}\ ,\ w \text{ is a subsolution}\}$$

is a solution. The comparison theorem is of course enough to guarantee uniqueness. The constructibility of $\bar{u}$ and $\bar{v}$ depends on the circumstances and is relatively easy to establish.

The richness of the theory is in its flexibility. One can prove stability results of various kinds and the validity of various approximation schemes. One can modify the theory to include Neumann boundary conditions. This is tricky because one has to interpret the normal derivative suitably for a function that has no smoothness requirements and the boundary condition can be nonlinear as well. Treating parabolic equations is not any different. One can just consider t as another variable.

I would suggest the survey article by Crandall, Ishii, and Lions that appeared in the Bulletin of the American Mathematical Society in 1992 for those who want to read more about this area.

The second body of work that I want to discuss has to do with the Boltzmann equation and similar equations. During the last six or seven years Pierre-Louis Lions has played the central role in new developments in the theory of the Boltzmann equation and similar transport equations. These are important in kinetic theory and arise in a wide variety of physical applications. We will for simplicity stay within the context of the Boltzmann equation. In R^3 we have a collection of particles moving along and interacting through "collisions" among themselves. As we do not want to keep track of the positions and velocities of the particles individually, we abstract the situation by the density $f(x, v)$ of particles that are at position x with velocity v. Even if there is no interaction, the density $f(x, v)$ will change in time due to uniform motion of the particles. The time-dependent density $f(t, x.v)$ will satisfy the equation

$$\frac{\partial f}{\partial t} + v.\nabla_x f = 0.$$

The collisions will change this equation to

$$\frac{\partial f}{\partial t} + v.\nabla_x f = Q(f,f).$$

Here Q is a quadratic quantity that represents binary collisions and its precise form depends on the nature of the interaction. Generally it looks like

$$Q(f,f) = \int\int_{R^3 \times S^2} dv_* \, d\omega B(v - v_*, \omega)\{f' f'_* - f f_*\}.$$

The notation here is standard: v and v_* are the incoming velocities and v' and v'_* are the outgoing velocities. B is the collision kernel. For given incoming velocities v and v_*, ω on the sphere S^2 parametrizes all the outgoing velocities compatible with the conservation of energy and momenta.

$$v' = v - (v - v_*, \omega)\omega, \qquad v'_* = v + (v - v_*, \omega)\omega$$

and f', f_*, f'_* are $f(t,x,v)$ with v replaced by the correspondingly changed v', v_*, and v'_*.

This problem of course has a long history. Smooth and unique solutions had been obtained for small time or globally for initial data close to equilibrium. Carleman had studied spatially homogeneous solutions. But a general global existence theorem had never been proved. The work of Lions (in collaboration with DiPerna) is a breakthrough for this and many other related transport problems of great physical interest.

Let me spend a few minutes giving you some idea of the method as developed by Lions and others (mostly his collaborators).

Although the nonlinearity looks somewhat benign it causes a serious problem in trying to establish any existence results. The collision term is quadratic and involves both positive and negative terms. To carry out any analysis one must control each piece separately. One gets certain a priori estimates from the conservation of mass and energy. The Boltzmann H-theorem gives an important additional control if one starts with an initial data with finite entropy. If we denote by Q^+ and Q^- the positive and negative terms in the collision term with considerable effort one is able to obtain only local L_1 bounds on $(1+f)^{-1}Q^{\pm}(f,f)$. The weak solutions are therefore formulated in terms of $\log(1+f)$. As there are no smoothness estimates in x one has to show that the velocity integrals contained in Q provide the compactification needed to make the weak limit behave properly.

This idea of "velocity averaging", which is central to these methods, is easy to state in a simple context. Suppose we have a function $g(x,v)$ in $R^N \times R^N$ and for some reasonable function $a(v)$ we have a local L_p estimate on $a(v).\nabla_x g(x,v)$. Then for a good test function ψ the velocity integral

$$\int_{R^N} \psi(v) g(x,v)\, dv$$

is in a suitable Sobolev space. Another important step that is needed in dealing with the Vlasov equation is the ability to integrate vector fields with minimal

regularity. In nonlinear problems you have to learn to live with the regularity that the problem gives you. The writeup by Lions in the Proceedings of the last ICM (Kyoto 1990) provides a survey with references.

The third and final topic that I would like to touch on is the contribution Lions has made to a class of variational problems. There are many nonlinear PDEs that are Euler equations for variational problems. The first step in solving such equations by the variational method is to show that the extremum is attained. This requires some coercivity or compactness. If the quantity to be minimized has an "energy"-like term involving derivatives, then one has control on local regularity along a minimizing sequence. This usually works if the domain is compact. If the domain is noncompact the situation is far from clear. Take for instance the problem of minimizing

$$\int_{R^N} |(\nabla f)(x)|^2 dx - \int\int V(x-y) f^2(x) f^2(y)\, dx\, dy$$

over functions f with L_2 norm λ (fixed positive number). Here $V(.)$ is a reasonable function decaying at ∞. Because of translation invariance, the minimizing sequence must be centered properly in order to have a chance of converging. The key idea is that, in some complicated but precise sense, if the minimizing sequence cannot be centered, then any member of the sequence can be thought of as two functions with supports very far away from each other. If we denote the infimum by $\sigma(\lambda)$, then along such sequences the infimum will be $\sigma(\lambda_1) + \sigma(\lambda_2)$ with $\lambda_1 + \lambda_2 = \lambda$ $0 < \lambda_1, \lambda_2 < \lambda$ rather than $\sigma(\lambda)$. If independently one can show that $\sigma(\lambda)$ is strictly subadditive, then one can prove the existence of a minimizer. This idea has been developed fully and applied successfully by Lions to many important and interesting problems.

See the papers in Annales de l'Institut Henri Poincaré, Analyse Non Linéaire 1984 by Lions for many examples where this point of view is successfully used.

References

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