Entropy Methods in Hydrodynamic Scaling

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1 Introduction

Hydrodynamic scaling is a procedure that attempts rigorously to derive large scale behavior of complex interacting systems from laws governing its evolution that are specified at a smaller scale. The procedure involves statistical averaging over the small scales and can be viewed as part of nonequilibrium statistical mechanics.

The basic example is the classical problem of starting with a Hamiltonian system of interacting particles and deriving from it, after rescaling, the Euler equations of compressible gas dynamics.

Let us consider a Hamiltonian system of N particles moving in a rather large physical space, for instance, the 3-dimensional torus of size ℓ. These particles are governed by a pair interaction $V(x-y)$ between particles. $V(\cdot)$ is an even function that is nonnegative and has support in a fixed compact set independent of ℓ. N and ℓ will be large with $N = \ell^3$ so that the interparticle distance is of order 1 and each particle will typically see only a few particles at any given time. The phase space is $(T_\ell^3 \times R^3)^N$ and the equations of motion for the positions and velocities $(x_i(t), u_i(t))$ are

$$\begin{cases} \dfrac{dx_i^\alpha(t)}{dt} = \dfrac{\partial H}{\partial u_i^\alpha} \\ \dfrac{du_i^\alpha(t)}{dt} = -\dfrac{\partial H}{\partial x_i^\alpha} \end{cases} \tag{1.1}$$

Here i is the particle number, $1 \le i \le N$, and α is the coordinate index, $\alpha = 1, 2, 3$. The Hamiltonian $H(x_1, \dots, x_N; u_1, \dots, u_N)$ is, of course, given by

$$H(x;u) = \frac{1}{2}\sum_i \|u_i\|^2 + \frac{1}{2}\sum_{i \ne j} V(x_i - x_j). \tag{1.2}$$

The system has five conserved quantities: the number of particles N, the momenta $\sum u_i^\alpha$, and the total energy H. Suppose we rescale the torus to have size 1 and rescale time by a similar factor ℓ, then quantities of the form

$$\frac{1}{N}\sum J\left(\frac{x_i(\ell t)}{\ell}\right) \tag{1.3}$$

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$$\frac{1}{N}\sum J\left(\frac{x_i(\ell t)}{\ell}\right) u_i^\alpha(\ell t) \tag{1.4}$$

and

$$\frac{1}{N}\sum J\left(\frac{x_i(\ell t)}{\ell}\right) e_i(\ell t) \tag{1.5}$$

where

$$e_i = \frac{1}{2}\|u_i\|^2 + \frac{1}{2}\sum_{j:j\neq i} V(x_i - x_j)$$

change in a reasonable manner in t. As N and $\ell \to \infty$ in the manner specified, one should think of (1.3), (1.4), and (1.5) as converging to

$$\int J(y)\rho(y,\tau)\, dy \tag{1.6}$$

$$\int J(y)v^\alpha(y,\tau)\rho(y,\tau)\, dy \tag{1.7}$$

and

$$\int J(y)e(y,\tau)\rho(y,\tau)\, dy. \tag{1.8}$$

Here y and τ are rescaled space and time. $\rho(y,\tau)$ is the density at a given point of space time. $\{v^\alpha(y,\tau)\}$ are the local fluid velocities and $e(y,\tau)$ is the energy density, related to the temperature at a given point. The equations of gas dynamics in this context are a system of symmetric hyperbolic conservation laws that one can write down for the evolution of ρ, v^α, and e. These equations are somewhat different from the usual Euler equation one derives from the Boltzmann equation. In the Boltzmann limit the real density is small; thus, the Euler equation derived from it is linearization in ρ of our equations.

This classical model is deterministic and any randomness has to be in the initial configuration. A precise formulation of the problem has to be done carefully. The goal is to establish some rigorous connection between the Hamiltonian equations on one hand and the Euler equations on the other. Randomness is important because some averaging has to be done with respect to small scales and one needs some information as to how the particles will arrange themselves locally in phase space if we only know their local density, local average velocity, and local temperature. One expects the arrangement to be given by an appropriate Maxwell-Gibbs distribution and formally the equations are derived under that assumption. To justify it, at the least, one needs a reasonable ergodic theory and for that more noise is better.

We will first look at two other models where the evolution is stochastic, say something about these models, and return at the end to our classical model.

The next example is referred to as simple exclusion. The physical space will be the periodic d-dimensional integral lattice of period ℓ. After scaling by a factor of ℓ, this will be viewed as living inside the unit d-torus. We will have a certain number $N = \rho\ell^d$ of particles located at some of the lattice sites. Each site can have at most one particle. A particle at the lattice site x waits for a random exponential waiting time with mean 1 and then picks a random new site x' to which it tries to jump. If the new site already has a particle, then the jump cannot be completed and our particle waits again for a new exponential random time. If the site x' is free, it jumps and starts afresh. The choice of x' is made with probability $\pi(x'-x)$. The probability distribution $\pi(z)$ is of jump sizes, is independent of ℓ, and is assumed to have finite support. All the particles are doing this at the same time and independently of each other. Because of continuous time there will be no ties to resolve.

The only conserved quantity here is the number of particles and we look at the density. We want to study the behavior of

$$\frac{1}{\ell^d}\sum_{I=1}^{M} J\left(\frac{x_i(\tau)}{\ell}\right) \tag{1.9}$$

where $\tau = \ell^\alpha t$. The rescaling of time can be with either $\alpha = 1$ or 2.

The mean $m = \sum z\pi(z)$ plays an important role. If $m \neq 0$, the motion is convective and one has to take $\alpha = 1$. If we think of (1.9) as converging to

$$\int J(y)\rho(\tau,y)\,dy \tag{1.10}$$

then ρ is expected to satisfy the scalar conservation law

$$\frac{\partial\rho}{\partial\rho} + \nabla\cdot m\rho(1-\rho) = 0. \tag{1.11}$$

If $m = 0$, then we need to take $\alpha = 2$. The scaling is diffusive and one then expects ρ to satisfy a nonlinear diffusion equation

$$\frac{\partial\rho}{\partial\tau} = \frac{1}{2}\nabla\cdot a(\rho(\tau,y)\nabla\rho). \tag{1.12}$$

If $\pi(z) = \pi(-z)$; i.e., π is symmetric, then the problem is rather easy and, in fact, one can verify that

$$a(\rho) \equiv a = \sum z\otimes z\ \pi(z) \tag{1.13}$$

is a constant matrix and is, in fact, the covariance matrix of $\pi(\cdot)$. The limiting equation in this case is the linear heat equation.

The third example we will consider is referred to as the Ginzburg-Landau model and is a lattice field model. We again start with the periodic lattice of size ℓ in d dimensions. We scale it down and think of it inside the unit torus. At each lattice site x, we have a variable $\xi(x,t)$ that is real valued and changes in time. The collection $\{\xi(x,t)\}$ is an ℓ^d-dimensional diffusion process and can be described either through a set of stochastic differential equations or though its infinitesimal generator. We will do the former. If $e_1,\ldots,e_d$ are the d positive coordinate directions, then $x\pm e_i$ are the neighbors of x.

$$d\xi(x,t) = \sum_{i=1}^{d}\left[d\eta(x-e_i,x,t) - d\eta(x,x+e_i,t)\right] \tag{1.14}$$

$$d\eta(x, x+e_i, t) = \frac{1}{2}\left[\phi'(\xi(x,t)) - \phi'(\xi(x+e_i,t))\right] dt + d\beta_{x,x+e_i}(t) \tag{1.15}$$

The equation (1.14) tells us that the way $\xi(x,t)$ changes is by "stuff" coming in or going out along the bonds. We orient the bonds using the positive coordinate directions and the net change is an algebraic sum. Equation (1.15) tells us that the flow along a bond is proportional to some nonlinear gradient modified by white noise. Here $\phi'(\xi)$ is a nonlinear function that satisfies some natural assumptions. We use ϕ', the derivative of $\phi(\xi)$, for convenience. If we again take $\tau = \ell^2 t$ and consider

$$\frac{1}{\ell^d}\sum J\left(\frac{x}{\ell}\right)\xi(x,\tau) \tag{1.16}$$

as an approximation of

$$\int J(y)m(y,\tau)\, dy \tag{1.17}$$

where $m(y,\tau)$ is the limiting "density" of "stuff", then $m(\cdot,\cdot)$ is supposed to satisfy an equation of the form

$$\frac{\partial m}{\partial t} = \frac{1}{2}\Delta\lambda(m)$$

where $\lambda(m)$ is to be determined in terms of ϕ.

We shall look at our three examples in some detail in our next three sections and end with some comments.

2 Ginzburg-Landau Model

It is convenient to write down the infinitesimal generator of our diffusion process on R^{ℓ^d}.

$$\begin{aligned}(\mathcal{L}_\ell F) =& \frac{\ell^2}{2}\sum_{i,x}\left(\frac{\partial}{\partial\xi(x+e_i)} - \frac{\partial}{\partial\xi(x)}\right)^2 F \\ &- \frac{\ell^2}{2}\sum_{i,x}\left[\phi'(\xi(x+e_i)) - \phi'(\xi(x))\right]\left[\frac{\partial}{\partial\xi(x+e_i)} - \frac{\partial}{\partial\xi(x)}\right] F.\end{aligned} \tag{2.1}$$

The factor ℓ^2 appears due to rescaling of time. The object we want to study is

$$G(t) = \frac{1}{\ell^d}\sum J\left(\frac{x}{\ell}\right)\xi(x,t). \tag{2.2}$$

Using the stochastic differential equations one can write

$$dG(t) = A(t)dt + dM(t) \tag{2.3}$$

where

$$\begin{aligned}A(t) &= \mathcal{L}_\ell\left[\frac{1}{\ell^d}\sum J\left(\frac{x}{\ell}\right)\xi(x)\right](t) \\ &\simeq \frac{1}{2\ell^d}\sum(\Delta J)\left(\frac{x}{\ell}\right)\phi'(\xi(x,t))\end{aligned} \tag{2.4}$$

and

$$dM(t) \simeq 0$$

because the Brownian noise terms cancel each other out by averaging. The difficulty now, which is typical of all of these problems, is that due to the nonlinearity of ϕ' the equations do not close and one has to represent

$$\frac{1}{\ell^d}\sum(\Delta J)\left(\frac{x}{\ell}\right)\phi'(\xi(x,t)) \tag{2.5}$$

in terms of $m(x,t)$ as $\ell \to \infty$. These are, of course, defined by (1.16) and (1.17). These require a knowledge of how $\xi(x,t)$ are "distributed" for a given value of "m". If we know this, then the volume average of ϕ' can be replaced by a mean value of ϕ' for a given m.

The evolution governed by $\mathcal{L}_\ell$ is reversible or symmetric with respect to the weight

$$e^{-\sum_x \phi(\xi(x))} = \Phi_\ell(\xi). \tag{2.6}$$

(We normalize ϕ so that $\int e^{-\phi(\xi)}\, d\xi = 1$.) However, the process is not ergodic because

$$m = \frac{1}{\ell^d}\sum_x \xi(x) \tag{2.7}$$

is conserved under our evolution. The conditional distribution $\nu_{m,\ell}(d\xi)$ of $\{\xi(x)\}$, given the average m, are the invariant ergodic pieces. By an "equivalence of ensembles" type theorem we can show

$$\nu_{m,\ell}(d\xi) \to \prod_x e^{-\phi_m(\xi(x))}d\xi(x)$$

and

$$\phi_m(\xi) = \phi(\xi) - \lambda(m)\xi + p(\lambda(m)).$$

Here

$$p(\lambda) = \log\int \exp\left[\lambda\xi - \phi(\xi)\right] d\xi$$

and $\lambda(m)$ solves

$$p'(\lambda(m)) = m$$

and equals

$$\lambda(m) = \frac{d}{dm}\sup_\lambda\left[\lambda m - p(\lambda)\right].$$

One finally verifies that

$$\int \phi'(\xi)e^{-\phi_m(\xi)}d\xi = \lambda(m).$$

This yields our equation

$$\frac{\partial m}{\partial t} = \frac{1}{2}\Delta\lambda(m). \tag{2.8}$$

In order to justify this one has to prove that averages of the form

$$\frac{1}{|B|}\sum_{x\in B}\phi'(\xi(x))$$

are nearly equal to

$$\lambda\left(\frac{1}{|B|}\sum_{x\in B}\xi(x)\right)$$

most of the time with probability nearly 1. The size of the block is important. It should be of size $\ell\epsilon$ with $\epsilon << 1$ but fixed. In [5] with Guo and Papanicolaou we developed a method for handling the problem by using entropy and entropy production as tools. We assume that initially the field $\{\xi(x,0)\}$ is random and is given by a density $f_\ell^0(\xi)\Phi_\ell(\xi)$ satisfying an entropy bound

$$\int f_\ell^0(\xi)\log f_\ell^0(\xi)\Phi_\ell(\xi)\,d\xi \le C\ell^d. \tag{2.9}$$

Such a bound is natural and is satisfied in most cases because one can think of C as the bound for average entropy per site. Then the distribution of $\{\xi(x,t)\}$ will have a density $f_\ell^t(\xi)$ satisfying

$$H_\ell(t) = \int f_\ell^t(\xi)\log f_\ell^t(\xi)\Phi_\ell(\xi)\,d\xi \le H_\ell(0)$$

and, in fact,

$$\frac{dH_\ell(t)}{dt} = -\frac{\ell^2}{2}\int\sum_{i,x}\left(\frac{\partial f_\ell^t}{\partial\xi(x+e_i)} - \frac{\partial f_\ell^t}{\partial\xi(x)}\right)^2\frac{1}{f_\ell^t}\Phi_\ell(\xi)\,d\xi\,dt. \tag{2.10}$$

Because $H_\ell(t) \ge 0$ one gets a trivial bound

$$\int_0^\infty\sum_{i,x}\int\left(\frac{\partial f_\ell^t}{\partial\xi(x+e_i)} - \frac{\partial f_\ell^t}{\partial\xi(x)}\right)^2\frac{1}{f_\ell^t}\Phi_\ell(\xi)\,d\xi\,dt \le 2C\ell^{d-2}.$$

If we fix a finite time T and consider

$$\overline{f}_\ell = \frac{1}{T}\int_0^T f_\ell^t\,dt,$$

by convexity

$$\int\sum_{i,x}\left(\frac{\partial\overline{f}_\ell}{\partial\xi(x+e_i)} - \frac{\partial\overline{f}_\ell}{\partial\xi(x)}\right)^2\frac{1}{\overline{f}_\ell}\Phi_\ell(\xi)\,d\xi \le C_T\ell^{d-2}.$$

We showed in [5] that the above estimate was sufficient to justify the averaging lemma and establish equation (2.8) rigorously. The following theorem was proved.

Assume $f_\ell^0(\xi)$ satisfies the bound (2.8). Assume further that for some density $m_0(y)$ and all smooth test functions $J(\cdot)$

$$\frac{1}{\ell^d}\sum J\left(\frac{x}{\ell}\right)\xi(x) \to \int J(y)m_0(y)\,dy$$

in probability under $f_\ell^0(\xi)\Phi_\ell(\xi)d\xi$. Then for any $t > 0$

$$\frac{1}{\ell^d}\sum J\left(\frac{x}{\ell}\right)\xi(x,t) \to \int J(y)m(t,y)\,dy$$

in probability, where $m(t, y)$ solves (2.8) with initial condition $m_0(y)$.

In [14] a modified approach was developed by Yau that has much wider applicability. One could guess that the density at time t should look like

$$g_\ell^t(\xi)\Phi_\ell(\xi) = \frac{1}{Z}\exp\left[\sum \lambda\left(t, \frac{x}{\ell}\right)\xi(x)\right]\Phi_\ell(\xi)$$

where $Z = \exp[\sum p\left(t, \frac{x}{\ell}\right)]$ and $\lambda(t,x) = \lambda(m(t,x))$ and m solves (2.8). In general, the density is $\Phi_\ell(\xi)f_\ell^t(\xi)$ where $f_\ell^t(\xi)$ is obtained by solving the Fokker-Planck equation and is not g_ℓ^t, even if we start off with $f_\ell^0(\xi) = g_\ell^0(\xi)$.

The question is: How close are they?

If we define the specific entropy $s(f_\ell^t, g_\ell^t)$ by

$$s(f_\ell^t, g_\ell^t) = \frac{1}{\ell^d}\int f_\ell^t(\xi)\log\frac{f_\ell^t(\xi)}{g_\ell^t(\xi)}\Phi_\ell(\xi)\,d\xi, \tag{2.11}$$

the theorem of Yau in [14] is that uniformly in $0 \le t \le T$,

$$\lim_{\ell\to\infty} s(f_\ell^t, g_\ell^t) = 0.$$

This is enough to justify the hydrodynamic scaling and arrive at the same theorem as in [5]. One assumes more initially but one obtains a stronger conclusion.

The model that we have studied is very special because it is a "gradient" model. While computing

$$dG(t) = d\left(\frac{1}{\ell^d}\sum J\left(\frac{x}{\ell}\right)\xi(x,t)\right)$$

we obtained

$$\begin{aligned}
&\mathcal{L}_\ell\left(\frac{1}{\ell^d}\sum J\left(\frac{x}{\ell}\right)\xi(x)\right)\\
&= -\frac{1}{2}\frac{\ell^2}{\ell^d}\sum\left[J\left(\frac{x+e_i}{\ell}\right) - J\left(\frac{x}{\ell}\right)\right][\phi'(\xi(x+e_i)) - \phi'(\xi(x))]\cdot\\
&\simeq \frac{1}{2\ell^d}\sum(\Delta J)\left(\frac{x}{\ell}\right)\phi'(\xi(x))
\end{aligned}$$

We were able to carry out summation by parts twice. The local "flux" turned out to be

$$\phi'(\xi(x+e)) - \phi'(\xi(x)),$$

which is a gradient and is amenable to another summation by parts. With that the global flux turned out to be

$$\frac{1}{2\ell^d}\sum(\Delta J)\left(\frac{x}{\ell}\right)\xi(x),$$

which is an order 1 quantity.

This is not true in general. To see this let us modify our evolution somewhat. Our operator corresponds to the following Dirichlet form relative to the weight $\Phi_\ell(\xi)$

$$\mathcal{D}_\ell(F) = \frac{\ell^2}{2}\int\sum_{x,i}\left(\frac{\partial F}{\partial\xi(x+e_i)} - \frac{\partial F}{\partial\xi(x)}\right)^2 \Phi_\ell(\xi)d\xi.$$

We modify it by

$$\widetilde{\mathcal{D}}_\ell(F) = \frac{\ell^2}{2}\int\sum_{x,i} a(\xi(x+e_i),\xi(x))\left(\frac{\partial F}{\partial\xi(x+e_i)} - \frac{\partial F}{\partial\xi(x)}\right)^2 \Phi_\ell(\xi)\,d\xi$$

where $a(\xi_1,\xi_2)$ is a smooth positive function bounded above and below. For simplicity, let us take $d=1$. The operator $\widetilde{\mathcal{L}}_\ell$ is given by

$$\begin{aligned}\widetilde{\mathcal{L}}_\ell F =& \frac{\ell^2}{2}\sum a\left(\xi(x+1),\xi(x)\right)\left(\frac{\partial}{\partial\xi(x+1)} - \frac{\partial}{\partial\xi(x)}\right)^2 F\\ &-\frac{\ell^2}{2}\sum W(\xi(a+1),\xi(x))\left(\frac{\partial}{\partial\xi(x+1)} - \frac{\partial}{\xi(x)}\right) F\end{aligned}$$

where

$$\begin{aligned}W(\xi_1,\xi_2) = &\, a(\xi_1,\xi_2)\left(\phi'(\xi_1) - \phi'(\xi_2)\right)\\ &+ a_2(\xi_1,\xi_2) - a_1(\xi_1,\xi_2)\end{aligned}$$

and

$$a_i(\xi_1,\xi_2) = \frac{\partial}{\partial\xi_i}a(\xi_1,\xi_2)$$

for $i = 1,2$.

Now, if we compute

$$\widetilde{\mathcal{L}}_\ell\left(\frac{1}{\ell}\sum J\left(\frac{x}{\ell}\right)\xi(x)\right)$$

we get

$$-\frac{1}{2}\sum J'\left(\frac{x}{\ell}\right)W\left(\xi(x+1),\xi(x)\right). \tag{2.12}$$

The term in (2.12) is a big term. The mean value of W is zero for any given value of m and so averaging produces a meaningless product of zero and infinity.

In [12] we developed a method for handling problems of this kind. We showed that there is a function $D(m)$ such that

$$W(\xi_1, \xi_2) - D(m)\,[\phi'(\xi_1) - \phi'(\xi_2)]$$

is negligible in a somewhat complicated but precise sense. With this modification the correct diffusion equation is

$$\begin{aligned} \frac{\partial m}{\partial t} &= \frac{1}{2}\frac{\partial}{\partial y} D(m(t,y))\frac{\partial}{\partial y}\lambda(m(t,y)) \\ &= \frac{1}{2}\frac{\partial}{\partial y}\widetilde{D}(m(t,y))\frac{\partial m}{\partial y} \\ \widetilde{D}(m) &= D(m)\lambda'(m). \end{aligned}$$

For $D(m)$ we provided a variational formula that replaces the traditional Green-Kubo formula. In the Green-Kubo formula the diffusion coefficient $D(m)$ is a space-time integral (sum) of the autocorrelation function in equilibrium, and this is not convenient. Our variational formula is much more convenient and is given explicitly by

$$D(m) = \inf_f E_m\{a(\xi_1, \xi_2)(1 - G_f)^2\}.$$

Here E_m refers to the expectation with respect to the infinite product measure

$$\exp[-\sum \phi_m(\xi(x))] \prod d\xi_x$$

and the infimum is taken over all tame test functions depending on a finite number of coordinates. For each such f, G_f is the gradient defined by

$$(G_f(\xi)) = \sum_x \left(\frac{\partial}{\partial \xi(2)} - \frac{\partial}{\partial \xi(1)}\right)\tau_x f$$

and τ_x is the canonical shift operator on the product space.

In principle one could try to carry out the method of Yau in [14] and one would have to work with a trial function of the form

$$g_N^t(\xi) \simeq \exp\left[\sum \lambda\left(t, \frac{x}{\ell}\right)\xi(x) + \frac{1}{\ell}\sum_x \psi(t, x, \xi)\right]$$

for $g_N^t(\xi)$. One has to choose a suitable corrector ψ in order to carry out the analysis. This has been done very recently by Funaki, Uchiyama, and Yau.

3 Simple Exclusion

We will now return to simple exclusion models. The state of the system can be described by a configuration $\eta = \{\eta(x)\}$, $x \in Z_\ell$, where Z_ℓ is the lattice of integers modulo ℓ. For simplicity, we have taken $d = 1$. $\eta(x) = 1$ if there is a particle at x and 0 otherwise. The generator can be written as

$$(\mathcal{L}_\ell F)(\eta) = \ell^\alpha \sum_{x,x'} \eta(x)(1-\eta(x'))\pi(x'-x)\left[F\left(\eta^{x,x'}\right) - F(\eta)\right]. \tag{3.1}$$

Here $\eta^{x,x'}$ refers to the new configuration obtained when the particle at x jumps to x'. α will be either 1 or 2, depending on the scaling used for time.

The case $\alpha = 1$.

$$\begin{aligned}\mathcal{L}_\ell\left[\frac{1}{\ell}\sum J\left(\frac{x}{\ell}\right)\eta(x)\right] &= \ell\sum\left[J\left(\frac{x'}{\ell}\right) - J\left(\frac{x}{\ell}\right)\right](x'-x)\eta(x)(1-\eta(x')) \\ &\simeq m\int J'(y)\rho(y,t)(1-\rho(y,t))\,dy\end{aligned}$$

where m is the mean $\sum x\ \pi(x)$ of the jump distribution π. The last step is justified because one expects the probability that a site x is occupied to be the local density $\rho(\frac{x}{\ell}, t)$, with occupancy of different sites being independent.
This leads to the equation

$$\frac{\partial\rho}{\partial t} + m(\rho(1-\rho))_y = 0 \tag{3.2}$$

for the density ρ. The method of [5] does not work here. The method of relative entropy contained in [14] will work, but needs the solution $\rho(t,x)$ to be smooth. It is known that for most initial data, sooner or later, discontinuities will develop and so the method applies only up to the first shock. There are other coupling methods that establish convergence to the correct weak solution of (3.2). See, for instance, Rezakhanlou [10] for the best results in the case of attractive dynamics.

The case $\alpha = 2$. In the symmetric case; i.e., $\pi(z) = \pi(-z)$, we always have $m = 0$ and we take $\alpha = 2$.

$$\begin{aligned}\mathcal{L}_\ell\frac{1}{\ell}\sum J\left(\frac{x}{\ell}\right)\eta(x) &= \frac{\ell^2}{\ell}\sum\left[J\left(\frac{x'}{\ell}\right) - J\left(\frac{x}{\ell}\right)\right]\eta(x)(1-\eta(x'))\pi(x'-x) \\ &= \frac{\ell^2}{2\ell}\sum\left(J\left(\frac{x'}{\ell}\right) - J\left(\frac{x}{\ell}\right)\right)[\eta(x)-\eta(x')]\,\pi(x'-x) \\ &\sim \frac{1}{2}\sum J''\left(\frac{x}{\ell}\right)\eta(x)\left(\sum z^2\pi(z)\right) \\ &= \frac{D}{2}\sum J''\left(\frac{x}{\ell}\right)\eta(x).\end{aligned}$$

This yields

$$\frac{\partial\rho}{\partial t} = \frac{D}{2}\rho_{xx} \tag{3.3}$$

with

$$D = \sum z^2 \pi(z).$$

The nonlinearity miraculously cancels out and the equations close. No averaging is needed.

If we change the problem by coloring the particles so that some are green and some are red, we can ask how the colors spread. One should compute

$$\mathcal{L}_\ell \left[\frac{1}{\ell} \sum A\left(\frac{x}{\ell}\right) \eta_g(x) + \frac{1}{\ell} \sum B\left(\frac{x}{\ell}\right) \eta_r(x) \right]$$

and proceed from there. Here $\eta_g(x) = 1$ if there is a green particle at x and similarly for $\eta_r(x)$. $\eta_g(x) + \eta_r(x) \leq 1$ due to exclusion. The dynamics is color blind. The analysis is hard because the system turns out to be nongradient. The method of [12] was applied to this situation by Quastel in [9] and he obtained an elliptic system for the pair $\rho_g(t, y)$ and $\rho_r(t, y)$. One first solves for

$$\rho(t, y) = \rho_g(t, y) + \rho_r(t, y)$$

by the heat equation (3.3). Then $\rho_g(t, y)$ is solved by

$$\frac{\partial \rho_g}{\partial t} = \frac{1}{2} \frac{\partial}{\partial y} S(\rho) \frac{\partial}{\partial y} \rho_g - (b(t, y)\rho_g)_y$$

where

$$b(t, y) = \frac{1}{2} \frac{(S(\rho) - D)\rho_y}{\rho}$$

is the pressure due to density gradient. $S(\rho)$ is the self diffusion coefficient determined in [7] and really depends on ρ, with $S(\rho) \to D$ as $\rho \to 0$ as $S(\rho) \to 0$ as $\rho \to 1$. The case $\pi(1) = \pi(-1) = \frac{1}{2}$ is special and $S(\rho) \equiv 0$ in that case.

The case where $\pi(z) \neq \pi(-z)$ but still $m = 0$ is more complex. This is nongradient and nonreversible. The methods of [12] and [9] have to be modified. This was carried out by Xu in [13] who established a limiting equation of the form

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \frac{\partial}{\partial y} D(\rho) \frac{\partial \rho}{\partial y}$$

with $D(\rho) \geq D = \sum z^2 \pi(z)$. In general, $D(\rho) > D$ for $0 < \rho < 1$ with $D(\rho) \to D$ as $\rho \to 0$ or 1.

Navier-Stokes corrections. Let us return to the case $m \neq 0$, but start very close to equilibrium. We start with a density $\rho(y) = \frac{1}{2} + \frac{1}{\ell} q_0(y)$; i.e.,

$$P[\eta(x) = 1] = \frac{1}{2} + \frac{1}{\ell} q_0\left(\frac{x}{\ell}\right).$$

Now we can rescale with $\alpha = 2$. Recently Esposito, Marra, and Yau [4] have shown that when the dimension $d \geq 3$, the empirical density at the rescaled time t is of the form $\frac{1}{2}+\frac{1}{\ell}q(t,\frac{x}{\ell})$ to within an accuracy $o(\frac{1}{\ell})$, and $q(t,y)$ is given as the solution of a Navier-Stokes type equation; i.e., Burgers' equation with viscosity.

4 Hamiltonian System

Let us return finally to our classical Hamiltonian system. If we start with a random initial configuration given by a density g_N^0 at time $t=0$, and let the configuration evolve by the Hamiltonian motion, one can obtain the density of the configuration at time t by solving the Liouville flow

$$\frac{\partial f_N}{\partial t} = \mathcal{L}_N f_N \qquad f_N = g_N^0 \text{ at } t=0. \tag{4.1}$$

If we are given functions $\rho(y), v^\alpha(y)$, and $T(y)$ on T^3 representing local density, average velocity, and temperature, assuming that those values avoid regions of possible phase transitions, we can associate a density $g_N(y_1,\ldots,y_N;u_1,\ldots,u_N)$ in the phase space that is a family of slowly varying Maxwell-Gibbs distributions strung together. If we calculate averages like $\frac{1}{N}\sum J(y_i)$ we get $\int J(y)\rho(y)\,dy$ in probability as $N\to\infty$. If we pick ρ, v^α, and T to be, as functions of t and y, a solution of the Euler equation that is quite smooth in some interval, we can use $\rho(t,y), v^\alpha(t,y)$, and $T(t,y)$ to construct a time dependent family $g_N(t,y_1,\ldots,y_N;u_1,\ldots u_N)$ of such densities.

We would like to establish that the density obtained by the Euler equation; i.e., g_N is close to the density f_N obtained by solving the Hamiltonian or equivalently the Liouville flow. An ideal theorem will say that as $N\to\infty$ the specific (per particle) relative entropy

$$s(f_N,g_N) = \frac{1}{N}\int \log\frac{f_N}{g_N}\cdot f_N \quad dy\,du \to 0. \tag{4.2}$$

We note that if the specific entropy were to tend to zero, by the usual large deviation estimates, the local density, velocities, and temperature would be the same for f_N and g_N. This would establish the hydrodynamic limit. But we cannot quite prove such a theorem. We have problems at two levels. First there is difficulty with large velocities. This can be overcome by changing the kinetic energy in the Hamiltonian to a function $\phi(u)$ with a bounded gradient instead of $\frac{1}{2}\|u\|^2$. (One choice that will work is the relativistic kinetic energy.) This is a technical point. The more serious problem is the hunger for noise. It is needed to establish some ergodicity. But only very little is needed. We put it in as an additional noisy exchange of velocities between pairs of particles that conserves momenta and energy. The strength of this noisy term is much smaller than the exchange of velocities provided by the Hamiltonian equations. This is then a small second order perturbation of the Liouville operator that does not destroy the conservation laws. The Euler equations are still the same. With (4.1) now replaced by a modified Fokker-Planck equation, one can establish (4.2). The details can be found in our work [8] with Olla and Yau . A key step is that whereas the noise is responsible for keeping the velocitiy distributions locally Maxwellian, for a Hamiltonian dynamics the positions are then shown to satisfy the necessary ergodic behavior with the correct Gibbs distributions.

5 Comments

We have limited our discussion essentially to work that uses entropy-related methods to the study of problems of hydrodynamic scaling. There is considerable work that uses other methods to study similar problems and we have not described them. The monographs [2] and [11] are excellent sources for a much wider collection of material.

We have also not discussed issues of large deviation. In some sense entropy-related methods are intimately related to techniques of large deviation theory and the two often go hand in hand. See, for instance, [6] and [2] for connections to the methods of [5]. As for Hamiltonian systems, there is the earlier work of Boldrighini, Dobrusin, and Sukov [1], which deals with the case of hard rods in one dimension with elastic collision.

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