Hidden Markov and State Space Models
Asymptotic Analysis of Exact and Approximate Methods for Prediction, Filtering, Smoothing and Statistical Inference

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Abstract

State space and hidden Markov models can both be subsumed under the same mathematical structure. On a suitable probability space $(\Omega, \mathcal{A}, P)$ are defined $(X_1, Y_1, X_2, Y_2, \ldots, X_i, Y_i, \ldots)$ a sequence of random “variables” taking values in a product space $\prod_{i=1}^{\infty} (\mathcal{U}_i \times \mathcal{V}_i)$ with an appropriate sigma field. The joint behavior under $P$ is that the $X_j$ are stationary Markovian and that given $(X_i, Y_i, \ldots)$ the $Y_j$ are independent and further that $Y_j$ is independent of all $X_i : i \neq j$ given $X_j$. If $\mathcal{H}$ is finite these are referred to as Hidden Markov models. The general case though focusing on $X$ Euclidean is referred to as state space models. Essentially we observe only the $Y$’s and want to infer statistical properties of the $X$’s given the $Y$’s. The fundamental problems of filtering, smoothing prediction are to give algorithms for computing exactly or approximately the conditional distribution of $X_t$ given $(Y_1, \ldots, Y_t)$ (Filtering), the conditional distribution of $X_t$ given $Y_1, \ldots, Y_T$, $T > t$ (Smoothing) and the conditional distribution of $X_{t+1}, \ldots, X_T$ given $Y_1, \ldots, Y_t$ (Prediction). If as is usually the case $P$ is unknown and is assumed to belong to a smooth parametric family of probabilities $\{P_\theta : \theta \in \mathbb{R}^d\}$, we face the further problem of efficiently estimating $\theta$ using $Y_1, \ldots, Y_T$ (computation of the likelihood, and maximum likelihood estimation, etc.).

State space models have long played an important role in signal processing. The Gaussian case can be treated algorithmically using the famous Kalman filter [6]. Similarly since the 1970s there has been extensive application of Hidden Markov models in speech recognition with prediction being the most important goal. The basic theoretical work here, in the case $X$ and $Y$ finite (small) providing both algorithms and asymptotic analysis for inference is that of Baum and colleagues [1]. During the last 30-40 years these general models have proved of great value in applications ranging from genomics to finance—see for example [7].

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Unless the $X, Y$ are jointly Gaussian or $X$ is finite and small the problem of calculating the distributions discussed and the likelihood exactly are numerically intractable and if $Y$ is not finite asymptotic analysis becomes much more difficult. Some new developments have been the construction of so-called “particle filters” (Monte Carlo type) methods for approximate calculation of these distributions (see Doucet et al. [4]) for instance and general asymptotic methods for analysis of statistical methods in HMM [2] and other authors.

We will discuss these methods and results in the light of exponential mixing properties of the conditional (posterior) distribution of $(X_1, X_2, \ldots)$ given $(Y_1, Y_2, \ldots)$ already noted by Baum and Petrie [1] and recent work of the authors Bickel, Ritov and Ryden [3], Del Moral and Jacod in [4], Douc and Matias [5].

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References


