Optimal Transport Maps in Monge-Kantorovich Problem

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Abstract

In the first part of the paper we briefly describe the classical problem, raised by Monge in 1781, of optimal transportation of mass. We discuss also Kantorovich’s weak solution of the problem, which leads to general existence results, to a dual formulation, and to necessary and sufficient optimality conditions.

In the second part we describe some recent progress on the problem of the existence of optimal transport maps. We show that in several cases optimal transport maps can be obtained by a singular perturbation technique based on the theory of $\Gamma$-convergence, which yields as a byproduct existence and stability results for classical Monge solutions.

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1. The optimal transport problem and its weak formulation

In 1781, G. Monge raised in [26] the problem of transporting a given distribution of matter (a pile of sand for instance) into another (an excavation for instance) in such a way that the work done is minimal. Denoting by $h_0, h_1 : \mathbb{R}^2 \to [0, +\infty)$ the Borel functions describing the initial and final distribution of matter, there is obviously a compatibility condition, that the total mass is the same:

$$\int_{\mathbb{R}^2} h_0(x) \, dx = \int_{\mathbb{R}^2} h_1(y) \, dy. \quad (1.1)$$

Assuming with no loss of generality that the total mass is 1, we say that a Borel map $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is a transport if a local version of the balance of mass condition

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holds, namely
\[
\int_{\psi^{-1}(E)} h_0(x) \, dx = \int_E h_1(y) \, dy \quad \text{for any } E \subset \mathbb{R}^2 \text{ Borel.} \tag{1.2}
\]

Then, the Monge problem consists in minimizing the work of transportation in the class of transports, i.e.
\[
\inf \left\{ \int_{\mathbb{R}^2} |\psi(x) - x| h_0(x) \, dx : \psi \text{ transport} \right\}. \tag{1.3}
\]

The Monge transport problem can be easily generalized in many directions, and all these generalizations have proved to be quite useful:

- General measurable spaces \( X, Y \), with measurable maps \( \psi : X \to Y \);
- General probability measures \( \mu \) in \( X \) and \( \nu \) in \( Y \). In this case the local balance of mass condition (1.2) reads as follows:
\[
\nu(E) = \mu(\psi^{-1}(E)) \quad \text{for any } E \subset Y \text{ measurable.} \tag{1.4}
\]

This means that the push-forward operator \( \psi_\# \) induced by \( \psi \), mapping probability measures in \( X \) into probability measures in \( Y \), maps \( \mu \) into \( \nu \).

- General cost functions: a measurable map \( c : X \times Y \to [0, +\infty] \). In this case the cost to be minimized is
\[
W(\psi) := \int_X c(x, \psi(x)) \, d\mu(x).
\]

Even in Euclidean spaces, the problem of existence of optimal transport maps is far from being trivial, mainly due to the non-linearity with respect to \( \psi \) of the condition \( \psi_\# \mu = \nu \). In particular the class of transports is not closed with respect to any reasonable weak topology. Furthermore, it is easy to build examples where the Monge problem is ill-posed simply because there is no transport map: this happens for instance when \( \mu \) is a Dirac mass and \( \nu \) is not a Dirac mass.

In order to overcome these difficulties, in 1942 L.V. Kantorovich proposed in [21] a notion of weak solution of the transport problem. He suggested to look for plans instead of transports, i.e. probability measures \( \gamma \) in \( X \times Y \) whose marginals are \( \mu \) and \( \nu \). Formally this means that \( \pi_X_\# \gamma = \mu \) and \( \pi_Y_\# \gamma = \nu \), where \( \pi_X : X \times Y \to X \) and \( \pi_Y : X \times Y \to Y \) are the canonical projections. Denoting by \( \Pi(\mu, \nu) \) the class of plans, he wrote the following minimization problem
\[
\min \left\{ \int_{X \times Y} c(x, y) \, d\gamma : \gamma \in \Pi(\mu, \nu) \right\}. \tag{1.5}
\]

Notice that \( \Pi(\mu, \nu) \) is not empty, as the product \( \mu \otimes \nu \) has \( \mu \) and \( \nu \) as marginals. Due to the convexity of the new constraint \( \gamma \in \Pi(\mu, \nu) \) it turns out that weak topologies can be effectively used to provide existence of solutions to (1.5): this happens for instance whenever \( X \) and \( Y \) are Polish spaces and \( c \) is lower semicontinuous (see for instance [28]). Notice also that, by convexity of the energy, the infimum is attained on a extremal element of \( \Pi(\mu, \nu) \).
The connection between the Kantorovich formulation of the transport problem and Monge’s original one can be seen noticing that any transport map \( \psi \) induces a planning \( \gamma \), defined by \( (Id \times \psi) \# \mu \). This planning is concentrated on the graph of \( \psi \) in \( X \times Y \) and it is easy to show that the converse holds, i.e. whenever \( \gamma \) is concentrated on a graph, then \( \gamma \) is induced by a transport map. Since any transport induces a planning with the same cost, it turns out that
\[
\inf (1.3) \geq \min (1.5).
\]
Moreover, by approximating any plan by plans induced by transports, it can be shown that equality holds under fairly general assumptions (see for instance [3]). Therefore we can really consider the Kantorovich formulation of the transport problem as a weak formulation of the original problem.

If all extremal points of \( \Pi(\mu, \nu) \) were induced by transports one would get existence of transport maps directly from the Kantorovich formulation. It is not difficult to show that plannings \( \gamma \) induced by transports are extremal in \( \Pi(\mu, \nu) \). The converse holds in some very particular cases, but unfortunately it is not true in general. It turns out that the existence of optimal transport maps depends not only on the geometry of \( \Pi(\mu, \nu) \), but also (in a quite sensible way) on the choice of the cost function \( c \).

2. Existence of optimal transport maps

In this section we focus on the problem of the existence of optimal transport maps in the sense of Monge. Before discussing in detail in the next sections the two model cases in which the cost function is the square of a distance or a distance (we refer to [19] for the case of concave functions of the distance, not discussed here), it is better to give an informal description of the tools by now available for proving the existence of optimal transport maps.

**Strategy A** (Dual formulation). This strategy is based on the duality formula
\[
\min \text{(MK)} = \sup \left\{ \int_X h \, d\mu + \int_Y k \, dv \right\}
\]
where the supremum runs among all pairs \( (h, k) \in L^1(\mu) \times L^1(\nu) \) such that \( h(x) + k(y) \leq c(x, y) \). The duality approach to the (MK) problem was developed by Kantorovich, and then extended to more general cost functions (see [22]). The transport map is obtained from an optimal pair \( (h, k) \) in the dual formulation by making a first variation. This strategy for proving the existence of an optimal transport map goes back to the papers [18] and [11].

**Strategy B** (Cyclical monotonicity). In some situations the necessary (and sufficient) minimality conditions for the primal problem, based upon the so-called \( c \)-cyclical monotonicity ([32], [28], [29]) yield that any optimal Kantorovich solution \( \gamma \) is concentrated on a graph \( \Gamma \) (i.e. for \( \mu \)-a.e. \( x \) there exists a unique \( y \) such that \( (x, y) \in \Gamma \)) and therefore is induced by a transport \( \psi \).
This happens for instance when \( c(x, y) = H(x - y) \), with \( H \) strictly convex in \( \mathbb{R}^n \). This approach is pursued in the papers [19], [30].

**Strategy C (Singular perturbation with strictly convex costs).** One can try to get an optimal transport map by making the cost strictly convex through a perturbation and then passing to the limit (see [12] and Theorem 4.1, Theorem 4.2 below). The main difficulty is to show (strong) convergence at the level of the transport maps and not only at the level of transport plans.

**Strategy D (Reduction to a lower dimensional problem).** This strategy has been initiated by V.N.Sudakov in [33]. It consists in writing (typically through a disintegration) \( \mu \) and \( \nu \) as the superposition of measures concentrated on lower dimensional sets and in solving the lower dimensional transport problems, trying in the end to “glue” all the partial transport maps into a single transport map. This strategy is discussed in detail in [3] and used, together with a “variational” decomposition, in [5]. The simplest case is when the lower dimensional problems are 1-dimensional, since the solution of the 1-dimensional transport problem is simply given by an increasing rearrangement, at least for convex functions of the distance (see for instance [2], [28], [35]).

Strategies A and B are basically equivalent and yield existence and uniqueness at the same time: the first one could be preferable for someone, as a very small measure-theoretic apparatus is involved. On the other hand, it strongly depends on the existence of maximizing pairs in the dual formulation, and this existence issue can be more subtle than the existence issue for the primal problem (see [28] and the discussion in [3]). For this reason it seems that the second strategy can work for more general classes of cost functions.

Strategies C and D have been devised to deal with situations where the cost function is convex but not strictly convex. Also these two strategies are closely related, as the strictly convex perturbation often leads to an effective dimension reduction of the problem (see for instance [5]).

### 3. cost=distance

In this section we consider the case when \( X = Y \) and the cost function \( c \) is proportional to the square of a distance \( d \). For convenience we normalize \( c \) so that \( c = d^2/2 \). The first result in the Euclidean space \( \mathbb{R}^n \) has been discovered independently by many authors Y.Brenier [8], [9], S.T.Rachev and L.R.Rischendorf [27], [29], and C.Smith and M.Knott [31].

**Theorem 3.1** Assume that \( \mu \) is absolutely continuous with respect to \( \mathcal{L}^n \) and that \( \mu \) and \( \nu \) have finite second order moments. Then there exists a unique optimal transport map \( \psi \). Moreover \( \psi \) is the gradient of a convex function.

In this case the proof comes from the fact that both strategies A and B yield that the displacement \( x - \psi(x) \) is the gradient of a \( c \)-concave function, i.e. a function representable as

\[
h(x) = \inf_{(y, t) \in I} c(x, y) + t \quad \forall x \in \mathbb{R}^n
\]
for a suitable non-empty set $I \subset Y \times \mathbb{R}$. The concept of $c$-concavity [29] has been extensively used to develop a very general duality theory for the (MK) problem, based on (2.6). In this special Euclidean situation it is immediate to realize that $c$-concavity of $h$ is equivalent to concavity (in the classical sense) of $h - \frac{1}{2}|x|^2$, hence

$$\psi(x) = x - \nabla h(x) = \nabla \left[ \frac{1}{2}|x|^2 - h(x) \right]$$

is the gradient of a convex function. Finally, notice that the assumption on $\mu$ can be sharpened (see [19]), assuming for instance that $\mu(B) = 0$ whenever $B$ has finite $H^{n-1}$-measure. This is due to the fact that the non-differentiability set of a concave function is $\sigma$-finite with respect to $H^{n-1}$ (see for instance [1]). Also the assumption about second order moments can be relaxed, assuming only that the infimum of the (MK) problem with data $\mu, \nu$ is finite.

The following result, due to R. Mc Cann [25], is much more recent.

**Theorem 3.2** Assume that $M$ is a $C^3$, complete Riemannian manifold with no boundary and $d$ is the Riemannian distance. If $\mu, \nu$ have finite second order moments and $\mu$ is absolutely continuous with respect to $\text{vol}_M$ there exists a unique optimal transport map $\psi$.

Moreover there exists a $c$-concave potential $h : M \to \mathbb{R}$ such that

$$\psi(x) = \exp_x (-\nabla h(x)) \text{ vol}_M \text{-a.e.}$$

This Riemannian extension of Theorem 3.1 is non trivial, due to the fact that $d^2$ is not smooth in the large. The proof uses some semiconcavity estimates for $d^2$ and the fact that $d^2$ is $C^2$ for $x$ close to $y$ (this is where the $C^3$ assumption on $M$ is needed). It is interesting to notice that the results of [24] (where the eikonal equation is read in local coordinates), based on the theory of viscosity solutions — see in particular Theorem 5.3 of [23] — allow to push Mc Cann’s technique up to $C^2$ manifolds.

Can we go beyond Riemannian manifolds in the existence theory? A model case is given by stratified Carnot groups endowed with the Carnot-Carathéodory metric $d_{CC}$, as these spaces arise in a very natural way as limits of Riemannian manifolds with respect to the Gromov-Hausdorff convergence (see [20]). At this moment a general strategy is still missing, but some preliminary investigations in the Heisenberg group $H_n$ show that positive results analogous to the Riemannian ones can be expected. The following result is proved in [6]:

**Theorem 3.3** If $n = 1, 2$ and $\mu$ is a probability measure in $H_n$ absolutely continuous with respect to $L^{2n+1}$, then:

(a) there exists a unique optimal transport map $\psi$, deriving from a $c$-concave potential $h$;

(b) If $d_p \uparrow d_{CC}$ are Riemannian left invariant metrics then Mc Cann’s optimal transport maps $\psi_p$ relative to $c_p = d_p^2/2$ converge in measure to $\psi$ as $p \to \infty$.

The restriction to $H_n$, $n \leq 2$, arises from the fact that so far we have been able to carry on some explicit computations only for $n \leq 2$. We expect that this restriction could be removed. The proof of (b) is not direct, as Mc Cann’s exponential
representation $\psi_p = \exp_p^x(-\nabla^p h_p)$ “degenerates” as $p \to \infty$, because the injectivity radius of the approximating manifolds tends to 0. This is due to the fact that in CC metric spaces geodesics exist but are not unique, not even in the small. Finally, if we replace $c$ by the square of the Korányi norm (related to the fundamental solution of the Kohn sub-Laplacian), namely

$$\tilde{c}(x, y) := \frac{1}{2} \|y^{-1} x\|^2 \quad \text{with} \quad \| (z, t) \| := 4 \sqrt{|z|^4 + t^2}$$

(here we identify $H_n$ with $\mathbb{C}^n \times \mathbb{R}$) then we are still able to prove existence in any Heisenberg group $H_n$. The proof uses some fine properties of $BV$ functions on sub-Riemannian groups [4]. However, we can’t hope for a Riemannian approximation result, as the Korányi norm induces a metric $d_K$ which is not geodesic. It turns out that the geodesic metric associated to $d_K$ is a constant multiple of $d_{CC}$.

4. cost=distance

In this section we consider the case when $X = Y$ and the cost function $c$ is a distance. In this case both strategies A and B give only a partial information about the location of $y$, for given $x$. In particular it is not true that any optimal Kantorovich plan $\gamma$ is induced by a transport map. Indeed, if the first order moments of $\mu$ and $\nu$ are finite, the dual formulation provides us with a maximizing pair $(h, k) = (u, -u)$, with $u : X \to \mathbb{R}$ 1-Lipschitz. If $X = \mathbb{R}^n$ and the distance is induced by a norm $\| \cdot \|$, this provides the implication

$$(x, y) \in \text{spt} \gamma \quad \Rightarrow \quad y \in \left\{ x - s \xi : \xi \in (du(x))^*, s \geq 0 \right\} \quad (4.7)$$

at any differentiability point of $u$. Here we consider the natural duality map between covectors and vectors given by

$$L^* := \{ \xi \in \mathbb{R}^n : L(\xi) = \| L \|_* \quad \text{and} \quad \| \xi \| = 1 \}.$$ 

The most favourable case is when the norm is strictly convex (e.g. the Euclidean norm): in this situation the $*$ operator is single-valued and we recover from (4.7) an information on the direction of transportation, i.e. $(du(x))^*$, but not on the length of transportation. If the norm is not strictly convex (e.g. the $l_1$ or $l_\infty$ norm) then even the information on the direction of transportation, encoded in $(du(x))^*$, is partial.

The first attempt to bypass these difficulties came with the work of V.N. Sudakov [33], who claimed to have a solution for any distance cost function induced by a norm. Sudakov’s approach is based on a clever decomposition of the space $\mathbb{R}^n$ in affine regions with variable dimension where the Kantorovich dual potential $u$ associated to the transport problem is an affine function. His strategy is to solve the transport problem in any of these regions, eventually getting an optimal transport map just by gluing all these transport maps. An essential ingredient in his proof is Proposition 78, where he states that, if $\mu \ll \mathcal{L}^n$, then the conditional measures induced by the decomposition are absolutely continuous with respect to the Lebesgue
measure (of the correct dimension). However, it turns out that this property is not true in general even for the simplest decomposition, i.e. the decomposition in segments: G. Alberti, B. Kirchheim and D. Preiss found an example of a compact family of pairwise disjoint open segments in $\mathbb{R}^3$ such that the family $M$ of their midpoints has strictly positive Lebesgue measure (the construction is a variant of previous examples due to A.S. Besicovitch and D.G. Larman, see also [2] and [5]). In this case, choosing $\mu = \mathcal{L}^3 \setminus M$, the conditional measures induced by the decomposition are Dirac masses. Therefore it is clear that this kind of counterexamples should be ruled out by some kind of additional “regularity” property of the decomposition. In this way the Sudakov strategy would be fully rigorous. As noticed in [5], this regularity comes for free only in the case $n = 2$, using the fact that transport rays do not cross in their interior.

Several years later, L.C. Evans and W. Gangbo made a remarkable progress in [15], showing by differential methods the existence of a transport map, under the assumption that $\text{spt } \mu \cap \text{spt } \nu = \emptyset$, that the two measures are absolutely continuous with respect to $\mathcal{L}^n$ and that their densities are Lipschitz functions with compact support. The missing piece of information about the length of transportation is recovered by a $p$-laplacian approximation

$$-\text{div} \left( |\nabla u|^{p-2} \nabla u \right) = \mu - \nu, \quad u \in H^1_0(B_R), \quad R \gg 1$$

obtaining in the limit as $p \to +\infty$ a nonnegative function $\sigma \in L^\infty(\mathbb{R}^n)$ and a 1-Lipschitz function $u$ solving

$$-\text{div} \left( a \nabla u \right) = \mu - \nu, \quad |\nabla u| = 1 \mathcal{L}^n\text{-a.e. on } \{a > 0\}.$$

The diffusion coefficient $a$ in the PDE above plays a special role in the theory. Indeed, one can show (see [2]) that the measure $\sigma := a\mathcal{L}^n$, the so-called transport density, can be represented in several different ways, and in particular as

$$\sigma(B) = \int \mathcal{H}^1(B \cap [x,y]) \, d\gamma(x,y) \quad \forall B \subset \mathbb{R}^n \text{ Borel} \quad (4.8)$$

for some optimal planning $\gamma$. Notice that the total mass of $\sigma$ is $\int |x-y| \, d\gamma$, the total work done and the meaning of $\sigma(B)$ is the work done within $B$ during the transport process. This representation of the transport density has been introduced by G. Bouchitté and G. Buttazzo in [7], who showed that the a constant multiple of the transport density is a solution of their so-called mass optimization problem.

Later, in [2], it was shown that there is actually a 1-1 correspondence between solutions of the mass optimization problem and transport densities, defined as in (4.8).

One can also show ([2], [13], [16], [14]) that $\sigma$ is unique (unlike $\gamma$) if either $\mu$ or $\nu$ are absolutely continuous. Moreover, the nonlinear operator mapping $(\mu, \nu) \in L^1 \times L^1$ into $a \in L^1$ maps $L^p \times L^p$ into $L^p$ for $1 \leq p \leq \infty$.

Coming back to the problem of the existence of optimal transport maps with Euclidean distance $|x-y|$ (or, more generally, with a distance induced by a $C^2$ and uniformly convex norm), the first existence results for general absolutely continuous measures $\mu, \nu$ with compact support have been independently obtained by
L.Caffarelli, M.Feldman and R.Mc Cann in [12] and by N.Trudinger and L.Wang in [34]. Afterwards, the author established in [2] the existence of an optimal transport map assuming only that the initial measure \( \mu \) is absolutely continuous, and the results of [12] and [34] have been extended to a Riemannian setting in [17]. All these proofs involve basically a Sudakov decomposition in transport rays, but the technical implementation of the idea is different from paper to paper: for instance in [12] a local change of variable is made, so that transport rays become parallel and Fubini theorem, in place of abstract disintegration theorems for measures, can be used. The proof in [3], instead, uses the co-area formula to show that absolute continuity with respect to Lebesgue measure is stable under disintegration.

The following result [3] is a slight improvement of [12], where existence of an optimal transport map was established but not the stability property. The result holds under regularity and uniform convexity assumptions for the norm \( \| \cdot \| \).

**Theorem 4.1** Let \( \mu, \nu \) be with compact support, with \( \mu << \mathcal{L}^n \), and let \( \psi_\epsilon \) be the unique optimal transport maps relative to the costs \( c_\epsilon(x,y) := \|x-y\|^{1+\epsilon} \). Then \( \psi_\epsilon \) converge as \( \epsilon \downarrow 0 \) to an optimal transport map \( \psi \) for \( c(x,y) = \|x-y\| \).

The proof is based only the fact that any plan \( \gamma_0 \), limit of some sequence of plans \( (Id \times \psi_\epsilon) \), is not only optimal for the (MK) problem, but also for the secondary one

\[
\min_{\gamma \in \Pi_1(\mu,\nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x-y\| \ln(\|x-y\|) \, d\gamma, \tag{4.9}
\]

where \( \Pi_1(\mu,\nu) \) denotes the class of all optimal plannings for the Kantorovitch problem (the entropy function in (4.9) comes from the Taylor expansion of \( c_\epsilon \) around \( \epsilon = 0 \)). It turns out that this additional minimality property selects a unique plan induced by a transport \( \psi \) and, a posteriori, \( \psi \) is the same map built in [12]. A class of counterexamples built in [3] shows that the absolute continuity assumption on \( \mu \) cannot be weakened, unlike the strictly convex case.

This “variational” procedure seems to select extremal elements of \( \Pi(\mu,\nu) \) in a very effective way. This phenomenon is apparent in view of the following result [5], which holds for all “crystalline” norms \( \| \cdot \| \) (i.e. norms whose unit sphere is contained in finitely many hyperplanes).

**Theorem 4.2** Let \( \mu, \nu \) be as in Theorem 4.1 and let \( \psi_\epsilon \) be the unique optimal transport maps relative to the costs

\[
c_\epsilon(x,y) := \|x-y\| + \epsilon |x-y| + \epsilon^2 |x-y| \ln |x-y|.
\]

Then \( \psi_\epsilon \) converge as \( \epsilon \downarrow 0 \) to an optimal transport map \( \psi \) for \( c(x,y) = \|x-y\| \).

In this case a secondary and a ternary variational problem are involved, and we show that the latter has a unique solution which is also induced by a transport.

Some borderline cases between “crystalline” norms and “Euclidean” norms apparently can’t be attacked by any of the existing techniques. In particular the existence of optimal transport maps for the cost induced by a general norm in \( \mathbb{R}^n \), \( n \geq 3 \), is still open.
References
